Journal of Scientific Research and Studies Vol. 4(11), pp. 304-317, November, 2017
ISSN 2375-8791

http://www.modernrespub.org/jsrs/index.htm

Full Length Research Paper

On the 4th Clay millennium problem: Proof of the regularity of the solutions of the Euler and Navier-Stokes equations, based on the conservation of particles

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Accepted 30 October, 2017

The majority of the applications of fluid dynamics refer to fluids that during the flow the particles are conserved. It is natural to have in mind such physical applications when examining the 4th Clay Millennium Problem. The assumptions of the standard formulation of the above problem, although reflecting the finiteness and conservation of the momentum and energy, as well as the smoothness of incompressible physical flows, do not reflect the conservation of particles of the fluid as local structure. By formulating the later conservation law and adding it to the hypotheses, it becomes possible to prove the regularity both for the Euler and Navier-Stokes equations. From the physical point of view this may mean that: if a) the particles like neutrons, electrons and protons remain such particle during the flow or if atoms exist in the fluid that if b) the atoms remain atoms of the same atomic number during the flow or if there are molecules in the fluid, if c) the molecules remain molecules of the same chemical type during the flow, then the regularity (smoothness of flow at all times) for the 4th Clay Millennium problem is provable and holds. The methodology for such a proof is based on proving that if a Blow-up would exist then at least for a particle range, the total energy would also converge to infinite (see Propositions 5.1, 5.2) which is a contradiction to the hypothesis of finite initial energy of the standard formulation of the 4th Clay Millennium problem.

Key words: Incompressible flows, regularity, Navier-Stokes equations, 4th Clay millennium problem.

Mathematical Subject Classification: 76A02

1. Introduction

The famous problem of the 4th Clay Mathematical Institute as formulated by Fefferman (2006) is considered a significant challenge to the science of mathematical physics of fluids, not only because it has lasted the efforts of the scientific community for decades to prove it (or converses to it), but also because it is supposed to hide a significant missing perception about the nature of our mathematical formulations of physical flows through the Euler and Navier-Stokes equations.

When the 4th Clay Millennium Problem was officially formulated, the majority was hoping that the regularity was also valid in 3 dimensions as it had been proven to hold in 2 dimensions.

The main objective of this paper is to prove the regularity of the Navier-Stokes equations with initial data as in the official formulation of the 4th Clay Millennium Problem. However, we do so by adding a mathematical hypothesis, which from the physical point of view is obvious for standard physical flows of the laboratory. Discovering what physical aspect of standard flows is not captured by the mathematical hypotheses of the official formulation of the 4th Clay Millennium Problem, is believed to be also a most significant contribution to the science of mathematical physics in this area. Although the mathematical assumptions of the standard formulation reflect the finiteness and conservation of the momentum and energy and the smoothness of incompressible physical flows, they do not reflect the conservation of particles as a local structure in standard and normal flows.

By adding this physical aspect formulated simply in the context of continuous fluid mechanics, the expected result of regularity can be proved. The methodology for this, from the physical point of view is described in paragraph 2.

A short outline of the logical structure of the paper is as follows:

(1) Paragraph 3 contains the official formulation of the 4th Cay millennium problem as described by Fefferman (2006). The standard formulation is any one of the four different conjectures, in which two of them assert the existence of blow-up in the periodic and non-periodic case, as well as two of them asserting the non-existence of blow-up, which is global in time regularity in the periodic and non-periodic case. We concentrate on proof of regularity in the non-periodic case or conjecture (A) that is described in Equations 1-6 after adding the conservation of particles as a local structure. Paragraph 3 contains definitions and more modern symbolism introduced by Tao (2013). The current paper follows the formal and mathematical austerity standards that the standard formulation has set, along with relevant results that have been suggested by the standard formulation in the literature as in the book by Majda and Bertozzi (2002).

However, we also try not to lose the intuition of the physical interpretation, as we are in the area of mathematical physics rather than pure mathematics. The goal is that the reader, after reading a dozen of mathematical propositions and their proofs, must be able at the end to have simple physical intuition, why the conjecture (A) of the 4th Clay millennium problem together with the conservation of particles in the hypotheses, holds.

(2) Paragraph 4 contains some known theorems and results that are to be used in this paper, so that the reader does not search for them in the literature and can have a direct, at a glance, image of what holds and what is proven. The most important are a list of necessary and sufficient conditions of regularity (Propositions 4.5-4.10). The same paragraph also contains some well-known and very relevant results that are not directly used, but are there for a better understanding of the physics.

(3) Paragraph 5 contains the main idea that the conservation of particles during the flow can be approximately formulated in the context of continuous fluid mechanics, and this is the key missing concept of conservation that acts as a subcritical invariant; in other words, it blocks the self-similar concentration of energy and turbulence that would create a blowup. With this new invariant we prove the regularity in the case of 3 dimensions: Propositions 5.2.

(4) Paragraph 6 contains the idea of defining a measure of turbulence in the context of deterministic mechanics based on the total variation of the component functions or norms (Definition 6.1). A significant observation is also made that the smoothness of the solutions of the Euler and Navier-Stokes equations is not a general type of smoothness, but one that would deserve the name “homogeneous smoothness” (Remark 6.2).

According to Constantin (2007), “… The blowup problem for the Euler equations is a major open problem of PDE, a theory of far greater physical importance than the blow-up problem of the Navier-Stokes equation, which of course is known to non-specialists because of the Clay Millennium problem…”

Almost all of our proven propositions and, in particular, the regularity in paragraphs 4, 5 and 6 (in particular, Proposition 5.2) are stated not only for the Navier-Stokes, but also for the Euler equations.

2. The methodology and the physical idea behind the current proof of regularity: The ontology of the continuous fluid mechanics models versus the ontology of statistical mechanics, discrete models

All researchers discriminate between the physical reality and its natural physical ontology (e.g. atoms, fluids, etc.) from the mathematical ontology (e.g. sets, numbers, vector fields, etc.). Furthermore, the discrete ontology of the statistical mechanics models of fluid dynamics and the ontology of the models of continuous fluid dynamics are discriminated. (see e.g. Fakour M et al. 2015a, b, c, 2017a, b, c). In the ontology of physical reality, as best captured by statistical mechanics’ models, the problem of the global 3-dimensional regularity seems easier to solve. For example, it is known (See Proposition 4.9 and Proposition 4.11 of maximum Cauchy development, which is referred also in the standard formulation of the Clay millennium problem by Fefferman (2006)) that if the global 3D regularity does not hold, then the velocities become unbounded or tend in absolute value to infinite as time gets close to the finite Blow-up time. Now we know that a fluid consists from a finite number of atoms and molecules, which also have finite mass and with a lower bound in their size. If such a phenomenon (Blowup) would occur, it would mean that at least for one particle the kinetic energy increases in an unbounded way. However, from the assumptions (see paragraph 3), the initial energy is finite, so this could never happen. We conclude that the fluid is 3D globally in the regular time. Unfortunately, such an argument, although valid in statistical mechanics’ models (Muriel, 2000), is not valid in continuous fluid mechanics models, where there are no atoms or particles with a lower bound of finite mass, but only points with zero dimension, and only mass density. We must also note that this argument is not likely to be successful if the fluid is compressible. In fact, it has been proven that a blow-up can occur even with a smooth compact initial data support, in the case of compressible fluids. One of the reasons is that if there is no lower bound in the density of the fluid, even without violating the momentum and energy conservation, a density converging to zero may lead to the velocities of some points converging to infinity. Nevertheless, if we formulate in the context of continuous fluid mechanics the conservation of particles as a local structure (Definition 5.1) then we can derive a similar argument (see proof of Proposition 5.1) where if a Blowup occurs in finite time, then the kinetic energy of a small finite ball (called in Definition 5.1, particle-range) will become unbounded, which is again impossible, due to the hypotheses of finite initial energy and energy conservation.

In statistical mechanical models of incompressible flow, we have the realistic advantage of many finite particles, e.g. like balls B(x,r) with finite diameter, r. These particles as they flow in time remain particles of the same nature and size, and the velocities, inside them, remain approximately constant.

Because space and time dimensions in classical fluid dynamics go in order of smallness that are smaller and at least as small as the real physical molecules, atoms and particles of the fluids, this may suggest imposing too, such conditions resembling uniform continuity conditions. In the case of continuous fluid dynamics models, such natural conditions, emerging from the particle nature of material fluids, together with the energy conservation, incompressibility and momentum conservation, as law preserved in time, may derive the regularity of local smooth solutions of the Euler and Navier-Stokes equations. For each atom or material particle of a material fluid, we can assume around it a ball of fixed radius, called particle range depending on the size of the atom or particle, which covers the particle and a bit of the electromagnetic, gravitational or quantum vacuum field around it, that their velocities and space-time accelerations are affected by the motion of the molecule or particle. For example, in the case of water, we are speaking here for molecules of H_2O, which are estimated to have a diameter of 2.75 angstroms or 2.75*10^(-10) meters, and we may define as water molecule particle range, the balls $B(r_0)$ of radius r_0=3*10^(-10) meters around the water molecule. As the fluid flows, especially in our case here of incompressible fluids, the shape and size of the molecules do not change much. Thus, there are no significant differences in the velocities and space-time accelerations of parts of the molecule. Bounds δ_u δ_ω of such differences remain constant as the fluid flows. We may call this effect as the “principle of conservation of particles” as a local structure. This principle must be set in the same setting as the energy conservation and incompressibility together with the Navier-Stokes or Euler equations. Of course, if the fluid is about solar plasma matter, such a description would not apply. However, incompressibility is hardly a property of it. But if we are talking about incompressible fluids that the molecule is conserved as well as the atoms and the atoms do not change their atomic number (as in fusion or fission), then this principle is physically valid. The principle of conservation of particles as a local structure blocks the self-similarity effects of concentrating energy and turbulence in very small areas and thus creating a Blow-up. It is the missing invariant in the discussion of many researchers about supercritical, critical and subcritical invariants in the scale transformations of the solutions. The exact definition of the conservation of particles as a local structure is found in Definition 5.1 and is as follows:

Conservation of particles as local structure in a fluid

Let a smooth solution of the Euler or Navier-Stokes equations for incompressible fluids that exists in the time interval [0,T). We may assume initial data on all of
R^3 or only on a connected compact support V_0. For simplicity, let us concentrate only on the latter simpler case. Let us denote by F as the displacement transformation of the flow. Let us also denote g as the partial derivatives of 1st order in space and time; that is, $\left|\partial_x^a \partial_t^b u(x)\right|$, |α|=1, |b|<=1,and call them space-time accelerations. We say that there is “conservation of the particles in the interval [0,T) in” a homogenous derivatives setting, as a local structure of the solution if and only if:

There is a small radius r, and small constants δ_x, δ_u, δ_ω,>0 so that for all t in [0,T) there is a finite cover C_t (in the case of initial data on R^3, it is infinite cover, but finite on any compact subset) of V_t , from balls B(r) of radius r, called “ranges of the particles”, such that:

(1) For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||F(x_1)-F(x_2)||<=r+\delta_x$ for all t>=s in [0,T).
(2) For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||u(F(x_1))-u(F(x_2))||<=\delta_u$ for all t >=s in [0,T).
(3) For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||g(F(x_1))-g(F(x_2))||<=\delta_\omega$ for all t >=s in [0,T).

If we state the same conditions 1), 2), and 3) for all times t in [0,+∞), then we say that we have the “strong version” of the conservation of particles as local structure.

We prove in paragraph 5, Proposition 5.2, that indeed, by adding the above conservation of particles as local structure in the hypotheses of the official formulation of the 4th Clay Millennium problem, we solve it, in the sense of proving the regularity (global in time smoothness) of the locally in time smooth solutions that are known to exist.

In Table 1, the hypotheses and conclusions are

Table 1. Continuous fluid dynamics versus statistical mechanics of fluids

Comparison and mutual significance of different types of mathematical models for the 4th Clay problem (no external force)	Continuous fluid mechanics model	Statistical mechanics model
Smooth Schwartz initial conditions	Yes	Possible to impose
Finite initial energy	Yes	Yes
Conservation of the particles	***Yes (non-obvious formulation)***	***Yes (obvious formulation)***
Local smooth evolution in an initial finite time interval	Yes	Possible to derive
Emergence of a blow-up in finite time	Impossible to occur	Impossible to occur

compared both in continuous fluid mechanics models and statistical mechanics models of the 4th Clay millennium problem in its standard formulation together with the hypothesis of conservation of particles. It would be paradoxical that we would be able to prove the regularity in statistical mechanics and we would not be able to prove it in continuous fluid mechanics.

3. The official formulation of the Clay Mathematical Institute as 4TH Clay millennium conjecture of 3D regularity and some definitions

In this paragraph, we highlight the basic parts of the standard formulation of the 4th Clay millennium problem, together with some more modern symbolism, since 2006, by relevant researchers, like T. Tao (2013).

In this paper, I consider the conjecture (A) of Fefferman (2006) standard formulation of the 4th Clay millennium problem, which I identify throughout the paper as the 4th Clay millennium problem.

The Navier-Stokes equations are given by (by R we denote the field of the real numbers, v>0 is the viscosity coefficient)

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \Delta u_i \quad (x \varepsilon R^3,\ t>=0,\ n=3) \qquad (1)$$

$$divu = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad (x \varepsilon R^3,\ t>=0,\ n=3) \qquad (2)$$

with initial conditions $u(x,0)=u^0(x)$ $\mathbf{x\varepsilon R^3}$
and u_0 (x) C∞ divergence-free vector field on $\mathbf{R^3}$ (3)

$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator. The Euler equations are when v=0

For physically meaningful solutions, we want to make sure that $u^0(x)$ does not grow large as $|x| \to \infty$. This is set by defining $u^0(x)$ and is called in this paper as "Schwartz initial conditions"; in other words,

$$\left|\partial_x^a u^0(x)\right| \le C_{a,K}(1+|x|)^{-K} \text{ on } \mathbf{R^3} \text{ for any } \alpha \text{ and K} \qquad (4)$$

(Schwartz used such functions to define the space of Schwartz distributions). We accept as physical meaningful solutions only if it satisfies

$$p,\ u \in C^\infty(R^3 \times [0,\infty)) \qquad (5)$$

and

$$\int_{\Re^3} |u(x,t)|^2 dx < C \quad \text{for all t>=0 (Bounded or finite energy)} \qquad (6)$$

The conjecture (A) of the Clay Millennium problem (case of no external force, but homogeneous and regular velocities) claims that for the Navier-Stokes equations, v>0, n=3, with divergence-free, Schwartz initial velocities, there are for all times t>0, smooth velocity and pressure fields, that are solutions of the Navier-Stokes equations with bounded energy; in other words, satisfying Equations 1, 2, 3, 4, 5, and 6. It is stated in the same standard formulation of the Clay millennium problem by Fefferman (2006) that the conjecture (A) has been proven to hold locally. "…if the time-interval [0,∞), is replaced by a small time interval [0,T], with T depending on the initial data....". In other words, there is ∞>T>0, such that there is continuous and smooth solution $u(x,t) \in C^\infty(R^3 \times [0,T))$. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^∞

Following Tao (2013), we define some specific terminology, about the hypotheses of the Clay millennium problem that will be used in the next.

We must notice that the definitions below can apply also to the case of inviscid flows, satisfying the Euler equations.

Definition 3.1 (Smooth solutions to the Navier-Stokes system): *A smooth set of data* for the Navier-Stokes system up to time T is a triplet (u_0, f, T), where 0 < T < ∞ is a time, the initial velocity vector field $u_0 : R^3 \to R^3$ and the forcing term f : $[0, T] \times R^3 \to R^3$ are assumed to be smooth on R^3 and $[0, T] \times R^3$, respectively (thus, u_0 is infinitely differentiable in space, and f is infinitely differentiable in space time), and u_0 is furthermore required to be divergence-free:

$\nabla \cdot u_0 = 0$.

If f = 0, we say that the data is *homogeneous.*

In the proofs of the main conjecture, we will not consider any external force, thus the data will always be homogeneous. But we will state intermediate propositions with external forces. Next we are defining simple differentiability of the data by Sobolev spaces.

Definition 3.2: We define the H^1 norm (or enstrophy norm) H^1 (u_0, f, T) of the data to be the quantity

H^1 (u_0, f, T):= $\|u_0\|_{H^1_X(R^3)} + \|f\|_{L^\infty_t H^1_X(R^3)} < \infty$ and say that (u_0, f, T) *is H^1* if H^1 (u_0, f, T) < ∞.

Definition 3.3: We say that a *smooth set of data* (u_0, f, T) is *Schwartz* if, for all integers α, m, k ≥ 0, one has

$$\sup_{x\in R^3}(1+|x|)^k \left|\nabla_x^a u_0(x)\right| < \infty$$

and $\sup_{(t,x)\in[0,T]\times R^3} (1+|x|)^k \left|\nabla_x^a \partial_t^m f\ (x)\right| < \infty$

Thus, for instance, the solution or initial data having Schwartz property implies having the H^1 property.

Definition 3.4: A *smooth solution* to the Navier-Stokes system, or a *smooth solution* for short, is a quintuplet (u, p, u_0 , f, T), where (u_0, f, T) is a *smooth set of data*, and the velocity vector field u : $[0, T] \times R^3 \to R^3$ and pressure field p : $[0, T]\times R^3 \to R$ are smooth functions on $[0, T]\times R^3$ that obey the Navier-Stokes equation (Equation 1) but with external forcing term f,

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu\Delta u_i + f_i \quad (x\epsilon R^3 ,\ t>=0 ,\ n=3)$$

and also the incompressibility property (Equation 2) on all of $[0, T] \times R^3$, but also the initial condition $u(0, x) = u_0(x)$ for all $x \in R^3$

Definition 3.5: Similarly, we say that (u, p, u_0, f, T*) is H^1* if the associated data (u_0, f, T) *is H^1*, and in addition one has

$$\|u\|_{L^\infty_t H^1_X([0,T]\times R^3)} + \|u\|_{L^2_t H^2_X([0,T]\times R^3)} < \infty$$

We say that the solution is *incomplete in [0,T),* if it is defined only in [0,t] for every t<T.

We use here the notation of *mixed norms* (e.g. in Tao, 2013). That is, if $\|u\|_{H^k_x(\Omega)}$ is the classical Sobolev norm, of smooth function of a spatial domain Ω, $u:\Omega \to R$, I is a time interval and $\|u\|_{L^p_t(I)}$ is the classical L^p -norm, then the mixed norm is defined by

$$\|u\|_{L^p_t H^k_x(I\times\Omega)} := (\int_I \|u(t)\|^p_{H^k_x(\Omega)} dt)^{1/p}$$

and

$$\|u\|_{L^\infty_t H^k_x(I\times\Omega)} := ess \sup_{t\in I}\|u(t)\|_{H^k_x(\Omega)}$$

which is similar instead of the Sobolev norm for other norms of function spaces.

We also denote by $C^k_x(\Omega)$, for any natural number $k \geq 0$, the space of all k times continuously differentiable functions $u:\Omega \to R$, with finite the next norm

$$\|u\|_{C^k_x(\Omega)} := \sum_{j=0}^{k}\left\|\nabla^j u\right\|_{L^\infty_x(\Omega)}$$

We use also the next notation for *hybrid norms*. Given two normed spaces X, Y on the same domain (in either space or time), we endow their intersection $X \cap Y$ with the norm

$$\|u\|_{X\cap Y} := \|u\|_X + \|u\|_Y .$$

In particular, we will use the next notation for intersection functions spaces, and their hybrid norms.

$$X^k(I\times\Omega) := L^\infty_t H^k_x(I\times\Omega) \cap L^2_x H^{k+1}_x(I\times\Omega) .$$

We also use the *big O notation*, in the standard way; that is, X=O(Y) means $X \leq CY$ for some constant C. If the constant C depends on a parameter, s, we denote it by C_s and we write $X=O_s(Y)$. We denote the difference of two sets A, B by A\B. And we denote Euclidean balls by $B(a,r) := \{x \in R^3 : |x-a| \leq r\}$, where |x| is the Euclidean norm.

With the above terminology, the target Clay millennium conjecture in this paper can be restated as the next proposition

The 4th Clay millennium problem (Conjecture A; Global regularity for homogeneous Schwartz data). *Let* (u_0*, 0, T) be a homogeneous Schwartz set of data. Then there exists a smooth finite energy solution (u, p,* u_0*, 0, T) with the indicated data (note that it is for any T>0,* thus global in time).

4. Some known or directly derivable, useful results that will be used

In this paragraph, the study states some known theorems and results, that are to be used in this paper, so that the reader does not search for them in the literature and can have a direct, at a glance, image of what holds and what is proved. A review of this paragraph is as follows:

Propositions 4.1 and 4.2 are mainly about the uniqueness and existence locally of smooth solutions of the Navier-Stokes and Euler equations with smooth Schwartz initial data. Proposition 4.3 are necessary or sufficient or necessary and sufficient conditions of regularity (global in time smoothness) for the Euler equations without viscosity. Equations 8-15 are forms of the energy conservation and finiteness of the energy loss in viscosity or energy dissipation. Equations 16-18 relate quantities for the conditions of regularity. Proposition 4.4 is the equivalence of smooth Schwartz initial data with smooth compact support initial data for the formulation of the 4th Clay millennium problem. Propositions 4.5-4.9 are necessary and sufficient conditions for regularity, either for the Euler or Navier-Stokes equations, while Propositions 4.10 is a necessary and sufficient condition of regularity for only the Navier-Stokes with non-zero viscosity.

Subsequently, the study uses the basic local existence and uniqueness of smooth solutions to the Navier-Stokes (and Euler) equations, which is usually referred also as the well-posedness, as it corresponds to the existence and uniqueness of the physical reality causality of the flow. The theory of well-posedness for smooth solutions is summarized in an adequate form for this paper by the Theorem 5.4 in Tao (2013).

The study gives first the definition of “mild solution” as in page 9 of Tao (2013). Mild solutions must satisfy a condition on the pressure given by the velocities. Solutions of smooth initial Schwartz data are always mild, but the concept of mild solutions is a generalization to apply for non-fast decaying in space initial data, as the Schwartz data, but for which data we may want also to have local existence and uniqueness of solutions.

Definition 4.1: We define a *H^1 mild solution* (u, p, u_0, f, T) to be fields $u, f : [0, T] \times R^3 \rightarrow R^3$, $p : [0, T] \times R^3 \rightarrow R$, $u_0 : R^3 \rightarrow R^3$, with $0 < T < \infty$, obeying the regularity hypotheses

$$u_0 \in H^1_x(R^3)$$

$$f \in L^\infty_t H^1_x([0,T]\times R^3)$$

$$u \in L^\infty_t H^1_x \cap L^2_t H^2_x([0,T]\times R^3)$$

with the pressure p being given by (Poisson)

$$p = -\Delta^{-1}\partial_i\partial_j(u_i u_j) + \Delta^{-1}\nabla\cdot f \qquad (7)$$

(Here the summation conventions are used not to write the Greek big Sigma), which obey the incompressibility conditions (Equations 2 and 3), and satisfy the integral form of the Navier-Stokes equations

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}(-(u\cdot\nabla)u - \nabla p + f)(t')dt'$$

with initial conditions $u(x,0)=u^0(x)$ **.**

We notice that the definition holds also for the inviscid flows, satisfying the Euler equations. The viscosity coefficient here has been normalized to v=1.

In reviewing the local well-posedness theory of H^1 mild solutions, the next can be said. The content of the Theorem 5.4 in Tao (2013; that the study also states here for the convenience of the reader and from which we derive our Proposition 4.2) is largely standard (and in many cases, it has been improved by more powerful current well-posedness theory). The study mentions here for example, the relevant research by Prodi (1959) and Serrin (1963). The local existence theory follows from the work of Kato and Ponce (1988). The regularity of mild solutions follows from the work of Ladyzhenskaya (1967, 1969). There are now a number of advanced local well-posedness results at regularity, especially that of Koch and Tataru (2001).

There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions with different methods. As it is referred in Fefferman (2006), the study refers to the reader by Majda and Bertozzi (2002), page 104 Theorem 3.4.

The study states here for the convenience of the reader the summarizing Theorem 5.4 as in Tao (2013). It omits the part (v) of Lipchitz stability of the solutions from the statement of the theorem. It uses the standard O() notation here, x=O(y) meaning x<=cy for some absolute constant c. If the constant c depends on a parameter k, we set it as index of $O_k()$.

It is important to remark here that the existence and uniqueness results locally in time (well-posedness), hold also not only for the case of viscous flows following the Navier-Stokes equations, but also for the case of inviscid flows under the Euler equations. There are many other papers and authors that have proved the local existence and uniqueness of smooth solutions both for the Navier-Stokes and the Euler equation with the same methodology, where the value of the viscosity coefficient v=0, can as well be included. The reader is been referred to Majda and Bertozzi (2002), page 104 Theorem 3.4, paragraph 3.2.3, and paragraph 4.1 page 138.

Proposition 4.1: *(Local well-posedness in H^1). Let (u_0, f, T) be H^1 data.*

(i) (Strong solution) If (u, p, u_0, f, T) is an H^1 mild solution, then

$$u \in C_t^0 H_x^1([0,T] \times R^3)$$

(ii) (Local existence and regularity) If

$$(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant c > 0, then there exists a H^1 mild solution (u, p, u_0, f, T) with the indicated data, with

$$\|u\|_{X^k([0,T]\times R^3)} = O(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0,T]\times R^3)} = O_k(\|u_0\|_{H_X^k(R^3)}, \|f\|_{L_t^1 H_X^1(R^3)}, 1)$$

for each k>=1. In particular, one has local existence whenever T is sufficiently small, depending on the norm H^1(u_0, f, T).

(iii) (Uniqueness) There is at most one H^1 mild solution (u, p, u_0, f, T) with the indicated data.

(iv) (Regularity) If (u, p, u_0, f, T) is a H^1 mild solution, and (u_0, f, T) is (smooth) Schwartz data, then u and p is smooth solution; in fact, one has

$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0,T] \times R^3)$ *for all j, K >=0.*

For the proof of the above theorem, the reader is referred to Tao (2013) theorem 5.4, but also to the papers and books, of the above mentioned other authors.

Next, the study states the local existence and uniqueness of smooth solutions of the Navier-Stokes (and Euler) equations with smooth Schwartz initial conditions, which will be used in this paper, explicitly as a Proposition 4.2 here.

Proposition 4.2: Local existence and uniqueness of smooth solutions or smooth well posedness. *Let **$u_0(x)$**, **$p_0(x)$** be smooth and Schwartz initial data at t=0 of the Navier-Stokes (or Euler) equations, then there is a finite time interval [0,T] (in general depending on the above initial conditions) so that there is a unique smooth local in time solution of the Navier-Stokes (or Euler) equations*

$$u(x),\ p(x) \in C^\infty(R^3 \times [0,T])$$

Proof: We simply apply the Proposition 4.1 above and in particular, from the part (ii) and the assumption in the Proposition 4.2, that the initial data are smooth Schwartz, we get the local existence of H^1 mild solution (u, p, u_0, 0, T). From the part (iv), we get that it is also a smooth solution. From the part (iii), we get that it is unique.

As an alternative, we may apply the theorems in Majda and Bertozzi (2002), page 104 Theorem 3.4, paragraph 3.2.3, and paragraph 4.1 page 138, and get the local in time solution, then derive from the part (iv) of the Proposition 4.1 above, that they are also in the classical sense smooth. QED.

Remark 4.1: We remark here that the property of smooth Schwartz initial data, is not in general conserved in later times than t=0, of the smooth solution in the Navier-Stokes equations, because it is a very strong fast decaying property at spatially infinity. But for lower rank derivatives of the velocities (and vorticity), we have the (global and) local energy estimate, and (global and) local enstrophy estimate theorems that reduce the decaying of the solutions at later times than t=0, at spatially infinite to the decaying of the initial data at spatially infinite (See e.g. Tao (2013), Theorem 8.2 (Remark 8.7) and Theorem 10.1 (Remark 10.6).

Furthermore, in the same paper of formal formulation of the Clay millennium conjecture (Fefferman, 2006), it is stated that the 3D global regularity of such smooth solutions is controlled by the bounded accumulation in finite time intervals of the vorticity (Beale et al., 1984). The study states this also explicitly for the convenience of the reader, for smooth solutions of the Navier-Stokes equations with smooth Schwartz initial conditions, as the Proposition 4.6 When we say here bounded accumulation e.g. of the deformations D, on finite internals, we mean in the sense, for example, of the proposition 5.1, page 171 in the book of Majda and Bertozzi (2002), which is a definition designed to control the existence or not of finite blowup times. In other words, for any finite time interval, [0, T], there is a constant M such that

$$\int_0^t |D|_{L^\infty}(s)ds <= M$$

Ii is stated here for the convenience of the reader, a well-known proposition of equivalent necessary and sufficient conditions of existence globally in time of solutions of the Euler equations, as inviscid smooth flows. It is the proposition 5.1 in Majda and Bertozzi (2002), page 171.

The *stretching* is defined by

$S(x,t) =: D\xi \cdot \xi$ if $\xi \neq 0$ and $S(x,t) =: 0$ if $\xi = 0$ where $\xi =: \frac{\omega}{|\omega|}$, ω being the vortcity.

Proposition 4.3: *Equivalent physical conditions for potential singular solutions of the Euler equations. The following conditions are equivalent for smooth Schwartz initial data:*

(1) The time interval, [0, T^*) with $T^* < \infty$ is a maximal interval of smooth H^s existence of solutions for the 3D

Euler equations.

(2) The vorticity ω accumulates so rapidly in time that

$\int_0^t |\omega|_{L^\infty}(s)ds \to +\infty$ *as t tends to T**

(3) The deformation matrix D accumulates so rapidly in time that

$\int_0^t |D|_{L^\infty}(s)ds \to +\infty$ *as t tends to T**

*(4) The stretching factor S(**x**, t) accumulates so rapidly in time that*

$\int_0^t [\max_{x\in R^3} S(x,s)]ds \to +\infty$ *as t tends to T**

The next theorem establishes the equivalence of smooth connected compact support initial data with the smooth Schwartz initial data, for the homogeneous version of the 4th Clay Millennium problem. It can be stated either for local in time smooth solutions or global in time smooth solutions. The advantage assuming connected compact support smooth initial data is obvious, as this is preserved in time by smooth functions, and also integrations are easier when done on compact connected sets.

Proposition 4.4: (3D global smooth compact support non-homogeneous regularity implies 3D global smooth Schwartz homogeneous regularity) *If it holds that the incompressible viscous (following the Navier-Stokes equations) 3-dimensional local in time [0,T], finite energy, flow-solutions with smooth compact support (connected with smooth boundary) initial data of velocities and pressures (thus finite initial energy) and smooth compact support (the same connected support with smooth boundary) external forcing for all times t>0, exist also globally in time t>0 (are globally regular) then it also holds that the incompressible viscous (following the Navier-Stokes equations) 3-dimensional local in time [0,T], finite energy, flow-solutions with smooth Schwartz initial data of velocities and pressures (thus finite initial energy), exist also globally in time for all t>0 (are regular globally in time).*

(for a proof see Kyritsis, 2017, Proposition 6.4)

Remark 4.2 Finite initial energy and energy conservation equations: When we want to prove that the smoothness in the local in time solutions of the Euler or Navier-Stokes equations is conserved, and that they can be extended indefinitely in time, we usually apply a "reduction ad absurdum" argument: Let the maximum finite time T* and interval [0,T*) so that the local solution can be extended smoothly in it.. Then the time T* will be a blow-up time, and if we manage to extend smoothly the solutions on [0,T*]. Then there is no finite Blow-up time T* and the solutions holds in [0,+∞). Below are listed necessary and sufficient conditions for this extension to be possible. Obviously no smoothness assumption can be made for the time T*, as this is what must be proved. But we still can assume that at T*, the energy conservation and momentum conservation will hold even for a singularity at T*, as these are universal laws of nature, and the integrals that calculate them, do not require smooth functions but only integrable functions, that may have points of discontinuity.

A very well known form of the energy conservation equation and accumulative energy dissipation is as follows:

$$\frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx + \int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt = \frac{1}{2}\int_{R^3}\|u(x,0\|^2 dx \quad (8)$$

where

$$E(0) = \frac{1}{2}\int_{R^3}\|u(x,0)\|^2 dx \quad (9)$$

is the initial finite energy

$$E(T) = \frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx \quad (10)$$

is the final finite energy

and $$\Delta E = \int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt \quad (11)$$

is the accumulative finite energy dissipation from time 0 to time T, because of viscosity into internal heat of the fluid. For the Euler equations, it is zero. Obviously

ΔE<=E(0)>=E(T) (12)

The rate of energy dissipation is given by

$$\frac{dE}{dt}(t) = -v\int_{R^3}\|\nabla u\|^2 dx < 0 \quad (13)$$

(v, is the viscosity coefficient. See Majda and Bertozzi, 2002; Proposition 1.13, equation (1.80) pp. 28).

Remark 4.3: The next are three very useful inequalities for the unique local in time [0,T], smooth solutions u of the Euler and Navier-Stokes equations with smooth Schwartz initial data and finite initial energy (they hold for more general conditions on initial data, but we will not use that):

By $||.||_m$ we denote the Sobolev norm of order m. So if m=0, it is essentially the L_2-norm. By $||.||_{L\infty}$ we denote the supremum norm, u is the velocity, ω is the vorticity, and cm and c are constants.

(1) $$\|u(x,T)\|_m \leq \|u(x,0)\|_m \exp(\int_0^T c_m \|\nabla(u(x,t)\|_{L_\infty} dt) \quad (14)$$

(see e.g. Majda and Bertozzi, 2002, proof of Theorem 3.6 pp.117, equation (3.79))

(2) $$\|\omega(x,t)\|_0 \leq \|\omega(x,0)\|_0 \exp(c\int_0^T \|\nabla u(x,t)\|_{L_\infty} dt) \quad (15)$$

(see e.g. Majda and Bertozzi, 2002, proof of Theorem 3.6 pp.117, equation (3.80))

(3) $$\|\nabla u(x,t)\|_{L_\infty} \leq \|\nabla u(x,0)\|_0 \exp(\int_0^t \|\omega(x,s)\|_{L_\infty} ds) \quad (16)$$

(see e.g. Majda and Bertozzi, 2002, proof of Theorem 3.6 pp.118, last equation of the proof)

The next are a list of well-known necessary and sufficient conditions, for regularity (global in time existence and smoothness) of the solutions of Euler and Navier-Stokes equations, under the standard assumption in the 4th Clay Millennium problem of smooth Schwartz initial data, that after the theorem of Proposition 4.4 above can be formulated equivalently with smooth compact connected support data. We denote by T* the maximum Blow-up time (if it exists) that the local solution u(x,t) is smooth in [0,T*).

(1) Proposition 4.5 (Necessary and sufficient condition for regularity): *The local solution u(x,t), t in [0,T*) of the Euler or Navier-Stokes equations, with smooth Schwartz initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the* ***Sobolev norm ||u(x,t)||m*** *, m>=3/2+2 , remains bounded, by the same bound in all of [0,T*), then , there is no maximal Blow-up time T*, and the solution exists as smooth in [0,+∞)*

Remark 4.4 See e.g. Majda and Bertozzi, 2002, pp. 115, line 10 from below)

(2) Proposition 4.6 (Necessary and sufficient condition for regularity: Beale et al., 1984): *The local solution u(x,t), t in [0,T*) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the vorticity is bounded by M, accumulation in [0,T*]:*

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M \quad (17)$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.5: See e.g. Majda and Bertozzi, 2002, pp. 115, Theorem 3.6. Also page 171 Theorem 5.1 for the case of inviscid flows. See also Lemarie-Rieusset (2002). Conversely, if regularity holds, then in any interval from the smoothness in a compact connected set, the vorticity is supremum bounded. The above theorems in the book (Majda and Bertozzi, 2002) guarantee that the above conditions extend the local in time solution to global in time; that is, to solutions (u, p, u_0, f, T) which is H^1 mild solution, for any T. Then applying the part (iv) of the Proposition 4.1 above, we get that this solution is also smooth in the classical sense, for all T>0, thus globally in time smooth.

(3) Proposition 4.7 (Necessary and sufficient condition for regularity): *The local solution u(x,t), t in [0,T*) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the vorticity is bounded by M, supremum norm L∞ in [0,T*]:*

$$\|\omega(x,t)\|_{L_\infty} \leq M \quad \textit{for all t in [0,T*)} \quad (18)$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.6: Obviously if $\|\omega(x,t)\|_{L_\infty} \leq M$, then also the integral exists and is bounded: $\int_0^{T^*} \|\omega(x,t)\|_{L_\infty} dt \leq M_1$ and the previous Proposition 4.6 applies. Conversely, if regularity holds, then in any interval from smoothness in a compact connected set, the vorticity is supremum bounded.

(4) Proposition 4.8 (Necessary and sufficient condition for regularity): *The local solution u(x,t), t in [0,T*) of the Euler or Navier-Stokes equations, with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0, so that the space accelerations are bounded by M, in the supremum norm L∞ in [0,T*]:*

$$\|\nabla u(x,t)\|_{L_\infty} \leq M \quad \textit{for all t in [0,T*)} \quad (19)$$

Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞)*

Remark 4.7: Direct from the inequality (Equation14) and the application of the proposition 4.5. Conversely if

regularity holds, then in any finite time interval from smoothness, the accelerations are supremum bounded.

(5) Proposition 4.9 (Fefferman, 2006; Necessary and sufficient condition for regularity): *The local solution u(x,t), t in [0,T*) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the velocities ||u(x,t)|| do not get unbounded as t->T*.*
Then there is no maximal Blow-up time T, and the solution exists smooth in [0,+∞).*

Remark 4.8: This is mentioned in the official formulation of the 4th Clay Millennium problem Fefferman (2006) p. 2, line 1 from below: quote "...For the Navier-Stokes equations (v>0), if there is a solution with a finite blowup time T, then the velocities $u_i(x,t)$, 1<=i<=3 become unbounded near the blowup time." The converse-negation of this is that if the velocities remain bounded near the T*, then there is no Blowup at T* and the solution is regular or global in time smooth. Conversely of course, if regularity holds, then in any finite time interval, because of the smoothness, the velocities, in a compact set are supremum bounded.

The study did not find a dedicated such theorem in the books or papers that were studied, but since prof. C.L Fefferman, who wrote the official formulation of the 4th Clay Millennium problem, was careful to specify that is in the case of non-zero viscosity v>0, and not of the Euler equations as the other conditions, it was assumed that he is aware of a proof of it.

(6) Proposition 4.10. (Necessary condition for regularity): *Let us assume that the local solution u(x,t), t in [0,T*) of the Navier-Stokes equations with non-zero viscosity, and with smooth compact connected support initial data, can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), in other words that are regular, then the trajectories-paths length l(a,t) does not get unbounded as t->T*.*

Proof: Let us assume that the solution is regular. Then also for all finite time intervals [0,T], the velocities and the accelerations are bounded in the L_∞, supremum norm, and this holds along all trajectory-paths too. Then also the length of the trajectories, as they are given by the formula

$$l(a_0,T)=\int_0^T \|u(x(a_0,t)\|dt \qquad (20)$$

are also bounded and finite (see e.g. Apostol, 1974, Theorem 6.6, p.128 and Theorem 6.17, p. 135). Thus if at a trajectory the lengths becomes unbounded as t goes to T*, then there is a blow-up. QED.

Remark 4.9.: Similar results about the local smooth solutions, hold also for the non-homogeneous case with external forces which is nevertheless space-time smooth of bounded accumulation in finite time intervals. Thus, an alternative formulation to see that the velocities and their gradient, or in other words up to their 1st derivatives and the external forcing also up to the 1st derivatives, control the global in time existence is the next proposition (Tao, 2013 Corollary 5.8).

Proposition 4.11 (**Maximum Cauchy development):** *Let (u_0, f, T) be H^1 data. Then at least one of the following two statements holds:*

1) There exists a mild H^1 solution (u, p, u_0, f, T) in [0,T] ,with the given data.
2) There exists a blowup time 0 < T^*< T and an incomplete mild H^1 solution.
(u, p, u_0, f, T^*) up to time T^* in [0, T^*), defined as complete on every [0,t], t<T^* which blows up in the enstrophy H^1 norm in the sense that

$$\lim_{t\to T^*, t<T^*}\|u(x,t)\|_{H^1_x(R^3)}=+\infty$$

Remark 4.10: The term "almost smooth" is defined in (Tao, 2013), before Conjecture 1.13. The only thing that almost smooth solutions lack when compared to smooth solutions is a limited amount of time differentiability at the starting time t = 0;

The term *normalized pressure*, refers to the symmetry of the Euler and Navier-Stokes equations to substitute the pressure, with another that differs at a constant in space but variable in time measureable function. In particular, normalized pressure is one that satisfies Equation 7 except for a measurable at a constant in space but variable in time measureable function. It is proved in Tao (2013), at Lemma 4.1, that the pressure is normalizable (exists a normalized pressure) in almost smooth finite energy solutions, for almost all times. The viscosity coefficient in these theorems of the above Tao paper has been normalized to v=1.

5. Conservation of the particles as a local structure of fluids in the context of continuous fluid mechanics: proof of the regularity for fluids with conservation of particles as a local structure, and the hypotheses of the standard formulation of the 4th Clay millennium problem, for the Euler and Navier-Stokes equations

Remark 5.1 (Physical interpretation of the definition 5.1): The smoothness of the particle-trajectory mapping (or displacement transformation of the points), the smoothness of the velocity field and vorticity field, is a condition that involves statements in the orders of micro-scales of the fluid, larger, equal and also by far smaller than the size of material molecules, atoms and particles, from which it consists. This is something that we tend to forget in continuous mechanics, because continuous mechanics was formulated before the discovery of the existence of material atoms. On the other-hand, it is traditional to involve the atoms and particles of the fluid, mainly in mathematical models of statistical mechanics. Nevertheless, we may formulate properties of material fluids in the context of continuous fluid mechanics that reflect approximately properties and behavior in the flow of the material atoms. This is in particular the Definition 5.1. For every atom or material particle of a material fluid, we may assume around it a ball of fixed radius, called "particle range" depending on the size of the atom or particle that covers the particle and a little bit of the electromagnetic, gravitational or quantum vacuum field around it, in which their velocities and space-time accelerations are affected by the motion of the molecule or particle. For example, for the case water, we are speaking here for molecules of H_2O that are estimated to have a diameter of 2.75 angstroms or 2.75*10^(-10) meters, we may define as water molecule particle range the balls $B(r_0)$ of radius r_0 =3*10^(-10) meters around the water molecule. As the fluid flows, especially in our case here of incompressible fluids, the shape and size of the molecules do not change much; neither there are significant differences of the velocities and space-time accelerations of parts of the molecule. Bounds δ_u δ_ω of such differences remain constant as the fluid flows. We may call this effect as the "principle of conservation of particles" as a local structure**.** This principle must be posed in equal setting as the energy conservation and incompressibility together with the Navier-Stokes or Euler equations. Of course if the fluid is about solar plasma matter, such a description would not apply. Nevertheless, then incompressibility is hardly a property of it. But if we are talking about incompressible fluids that the molecule is conserved as well as the atoms and do not change atomic number (as e.g. in fusion or fission), then this principle is physically valid. The principle of conservation of particles as a local structure, blocks the self-similarity effects of concentrating the energy and turbulence in very small areas and creating thus a Blow-up. It is the missing invariant in the discussion of many researchers about supercritical, critical and subcritical invariants in scale transformations of the solutions.

The next Definition 5.1 formulates precisely and mathematically this principle for the case of incompressible fluids.

Definition 5.1. (Conservation of particles as local structure in a fluid): *Let a smooth solution of the Euler or Navier-Stokes equations for incompressible fluids that exists in the time interval [0,T). We may assume initial data on all of R^3 or only on a connected compact support V_0 . For simplicity, let us concentrate only on the latter simpler case. Let us denote by F the point trajectories mapping of the flow. Let us also denote by g the partial derivatives of 1st order in space and time, that is* $\left|\partial_x^a \partial_t^b u\ (x)\right|$, *|α|=1, |b|<=1, and call them space-time accelerations. We say that there is "conservation of the particles in the interval [0,T) in" a derivatives homogenous setting, as a local structure of the solution if and only if:*
There is a small radius r, and small constants δ_x, δ_u, δ_ω, >0 so that for all t in [0,T), there is a finite cover C_t (in the case of initial data on R^3, it has infinite cover, but finite on any compact subset) of V_t, from balls B(r) of radius r, called "ranges of the particles", such that:

(1) For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||F(x_1)-F(x_2)||<=r+\delta_x$ for all t>=s in [0,T).
(2) For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||u(F(x_1))-u(F(x_2))||<=\delta_u$ for all t >=s in [0,T).
(3) For an x_1 and x_2 in a ball B(r) of V_s, s in [0,T), $||g(F(x_1))-g(F(x_2))||<=\delta_\omega$ for all t >=s in [0,T).

If we state the same conditions 1), 2), and 3) for all times t in [0,+∞), then we say that we have the "strong version" of the conservation of particles as local structure.

Proposition 5.1 (**Velocities on trajectories in finite time intervals with finite total variation, and bounded in the supremun norm uniformly in time):** *Let $u_t : V(t) -> R^3$ be smooth local in time in [0,T*), velocity fields solutions of the Navier-Stokes or Euler equations, with compact connected support V(0) initial data, finite initial energy E(0) and conservation of particles in [0,T*) as a local structure. The [0,T*) is the maximal interval that the solutions are smooth. Then for t in [0,T*)and x in V(t), the velocities are uniformly in time bounded in the supremum norm by a bound M independent of time t.*

$$\|u(x,t)\|_{L_\infty} = \sup_{x\in V(t)} \|u(x,t)\| \le M \quad \textit{for all t in [0,T*).}$$

Therefore, the velocities on the trajectory paths, in finite time intervals are of bounded variation and the trajectories in finite time interval, have finite length.

1st Proof (Only for the Navier-Stokes Equations): Let us assume that the velocities are unbounded in the supremum norm, as t converges to T*. Then there is a sequence of times t_n with t_n converging to time T*, and sequence of corresponding points $x_n(t_n)$, for which the norms of the velocities $||u(x_n(t_n), t_n)||$ converge to infinite.

$$\lim_{n\to+\infty} \| u(x(x_n, t_n), t_n) \| = +\infty \quad (21)$$

From the hypothesis of the conservation of particles as a local structure of the smooth solution in [0,T*), for every t_n, there is a finite cover C_{tn} of particle ranges, of V_{tn} so that $x_n(t_n)$ belongs to one such ball or particle-range $B_n(r)$ and for any other point $y(t_n)$ of $B_n(r)$, it holds that $||u(x_n(t_n), t_n)-u(y(t_n),t_n)|| <= \delta_u$. Therefore

$$||u(x_n(t_n), t_n)|| - \delta_u <= ||u(y(t_n),t_n)|| <= ||u(x_n(t_n), t_n)|| + \delta_u \quad (22)$$

for all times t_n in [0,T*).

By integrating spatially on the ball $B_n(r)$, and taking the limit as n->+∞ we deduce that

$$\lim_{n\to+\infty} \int_{B_n} \|u\| dx = +\infty$$

But this also means as we realize easily, that also

$$\lim_{n\to+\infty} \int_{B_n} \|u\|^2 dx = +\infty \quad (23)$$

Which nevertheless means that the total kinetic energy of this small, but finite and of constant radius, ball, converges to infinite, as t_n converges to T*. This is impossible by the finiteness of the initial energy, and the conservation of energy. Thus, the velocities are bounded uniformly, in the supremum norm, in the time interval [0,T*).

Therefore, the velocities on the trajectory paths, are also bounded in the supremum norm, uniformly in the time interval [0,T*). But this means by Proposition 4.9 that the local smooth solution is regular, and globally in time smooth, which from Proposition 4.8 means that the Jacobian of the 1st order derivatives of the velocities are also bounded in the supremum norm uniformly in time bounded in [0,T*). Which in its turn gives that the velocities are of bounded variation on the trajectory paths (see e.g. Apostol, 1974, Theorem 6.6, p. 128 and Theorem 6.17, p. 135) and that the trajectories have also finite length in [0,T*), because the trajectory length is given by the formula $l(a_0, T) = \int_0^T \|u(x(a_0, t)\| dt$. QED.

2nd Proof (Both for the Euler and Navier-Stokes equations): Instead of utilizing the condition (2) of the definition 5.1, we may utilize the condition (3). And we start assuming that the Jacobian of the velocities is unbounded in the supremum norm (instead of the velocities), as time goes to the Blow-up time T*. Similarly, we conclude that the energy dissipation density at a time on balls that are particle-ranges goes to infinite, giving the same for the total accumulative in time energy dissipation (Equation 11), which again is impossible from the finiteness of the initial energy and energy conservation. Then by Proposition 4.8, we conclude that the solution is regular, and thus also that the velocities are bounded in the supremum norm, in all finite time intervals. Again we deduce in the same way, that the total variation of the velocities is finite in finite time intervals and so are the lengths of the trajectories too. QED.

Proposition 5.2 (Global regularity as in the 4th Clay Millennium problem): *Let the Navier-Stokes or Euler equations with smooth compact connected initial data, finite initial energy and conservation of particles as local structure. Then the unique local in time solutions are also regular (are smooth globally in time).*

Proof: We apply the Proposition 5.1 above and the necessary and sufficient condition for regularity in Proposition 4.9 (which is only for the Navier-Stokes equations). Furthermore, we apply the part of the 2nd proof of the Proposition 5.1, which concludes regularity from Proposition 4.8, which holds for both the Euler and Navier-Stokes equations. QED.

6. Bounds of measures of the turbulence from length of the trajectory paths, and the total variation of the velocities, space acceleration and vorticity: the concept of homogeneous smoothness

Remark 6.1: In the next, we define a measure of the turbulence of the trajectories, of the velocities, of space-time accelerations and of the vorticity, through the "total variation" of the component functions in finite time intervals. This is in the context of deterministic fluid dynamics and not stochastic fluid dynamics. We remark that in the case of a blowup, the measures of turbulence below will become infinite.

Definition 6.1 (The variation measure of turbulence): Let smooth local in time in [0,T] solutions of the Euler or Navier-Stokes equations. The total length L(P) of a trajectory path P, in the time interval [0,T] is defined as the variation measure of turbulence of the displacements on the trajectory P, in [0,T]. The total variation TV(||u||) of the norm of the velocity ||u|| on the trajectory P in [0,T] is defined as the variation measure of turbulence of the velocity on the trajectory P in [0,T]. The total variation

TV(g) of the space-accelerations g (as in Definition 5.1) on the trajectory P in [0,T] is defined as the variation measure of turbulence of the space-time accelerations on the trajectory P in [0,T]. The total variation TV(||ω||) of the norm of the vorticity ||ω|| on the trajectory P in [0,T] is defined as the variation measure of turbulence of the vorticity on the trajectory P in [0,T].

Proposition 6.1 (Conservation in time of the boundedness of the maximum turbulence, which depends only on the initial data and time lapsed): *Let the Euler or Navier-Stokes equations with smooth compact connected initial data of finite initial energy and conservation of the particles as a local structure. Then for all times t, there are bounds $M_1 (t)$, $M_2 (t)$, $M_3(t)$, so that the maximum turbulence of the trajectory paths, of the velocities and of the space accelerations are bounded respectively by the above universal bounds, that depend only on the initial data and the time lapsed.*

Proof: From the Propositions 5.1 and 5.2, we deduce that the local in time smooth solutions are smooth for all times as they are regular. Then in any time interval [0,T], the solutions are smooth, and thus from the Proposition 4.8, the space acceleration g, are bounded in [0,T], thus also as smooth functions their total variation TV(g) is finite, and bounded (Apostol, 1974, Theorem 6.6, p. 128 and Theorem 6.17, p, 135). From the Proposition 4.7, the vorticity is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation TV(||ω||) is finite, and bounded on the trajectories. From the Proposition 4.9, the velocity is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation TV(||u||) is finite, and bounded on the trajectories. From the Proposition 4.10, the motion on trajectories is smooth and bounded in [0,T], thus also as smooth bounded functions its total variation, which is the length of the trajectory path L(P) is finite, and bounded in [0,T]. In the previous theorems, the bounds that we may denote them here by $M_1 (t)$, $M_2 (t)$, $M_3(t)$, respectively as in the statement of the current theorem, depend on the initial data, and the time interval [0,T]. QED.

Remark 6.2. (Homogeneity of smoothness relative to a property P): There are many researchers that they consider that the local smooth solutions of the Euler or Navier-Stokes equations with smooth Schwartz initial data and finite initial energy (even without the hypothesis of conservation of particles as a local structure) are general smooth functions. But it is not so! They are special smooth functions with the remarkable property that there are some critical properties P_i that if such a property holds in the time interval [0,T) for the coordinate partial space-derivatives of 0, 1, or 2 order, then this property holds also for the other two orders of derivatives. In other words, if it holds for the 2-order then it holds for the orders 0, 1 in [0,T). If it holds for the order 1, then it holds for the orders 0, 2 in [0,T]. If it holds for the order 0, then it holds also for the orders 1, 2 in [0,T]. This pattern for example, can be observed for the property P_1 of uniform boundedness in the supremum norm, in the interval [0,T*) in the Propositions 4.5-4.10. But one might try to prove it also for a second property P_2 which is the "finiteness of the total variation" of the coordinates of the partial derivatives, or even other properties P3 like "local in time Lipchitz conditions". This creates a strong bond or coherence among the derivatives and might be called *homogeneous smoothness relative to a property P.* We may also notice that the formulation of the conservation of particles as local structure is in such a way that as a property, it shows the same pattern of homogeneity of smoothness relative to the property of uniform in time bounds P_4 ,1), 2), 3) in the Definition 5.1. It seems to the study though that even this strong type of smoothness is not enough to derive the regularity, unless the homogeneity of smoothness is relative to the property P_4 ; in other words, the conservation of particles as a local structure.

7. Conclusions

The study believes that the main reasons of the failure so far in proving of the 3D global regularity of incompressible flows, with reasonably smooth initial conditions like smooth Schwartz initial data, and finite initial energy, as in the standard formal statement as the 4th Clay Millennium problem, is hidden in the difference of the physical reality ontology that is closer to the ontology of statistical mechanics models and the ontology of the mathematical models of continuous fluid dynamics.

Although energy and momentum conservation and finiteness of the initial energy are easy to formulate in both types of models, the conservation of particles as type and size is traditionally formulated only in the context of statistical mechanics. By succeeding in formulating approximately in the context of the ontology of continuous fluid mechanics the conservation of particles during the flow, as local structure, as in Definition 5.1, it becomes directly possible to prove the regularity in the case of 3 dimensions which is what most mathematicians were hoping that it should hold. In other words, and from the physical point of view, if: a) the particles like neutrons, electrons and protons remain such particle during the flow or if atoms exist in the fluid b) the atoms remain atoms of the same atomic number during the flow or if there are molecules in the fluid c) the molecules remain molecules of the same chemical type during the flow, then the regularity (smoothness of flow for all times) is provable and holds. The methodology for such a proof is based on proving that if a Blow-up would exist then at least for a particle range, the total energy would also converge to infinite (see Propositions 5.1, 5.2) which is a contradiction to the hypothesis of finite initial energy of the standard formulation of the 4th Clay Millennium problem.

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