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Error estimation of the differential Tau Method for certain fourth order boundary value problems

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In this paper, a generalized differential formulation of the Lanczos Tau Method and constructed polynomial error approximant of the error function for certain fourth order boundary value problems with first degree over-determination in ordinary differential equations were investigated; it is based on a modification of the error of Lanczos economization process. For this purpose, an algebraic linear system of equations was obtained by equating the corresponding coefficients of various powers of independent variable, in which these were solved to obtain the unknown constants. For the error estimation, two forms of homogeneous conditions of the error function were considered. Numerical experiments were given to illustrate the effectiveness of the method.

Key words: Chebyshev polynomial, differential operator, differential systems, error function, over-determination, Tau approximant, Tau parameter.

INTRODUCTION

Cornelius Lanczos remarks that an accurate approximate polynomial solutions of linear differential equations (ODEs) with polynomial coefficients defined in a finite interval $a \leq x \leq b$ can be obtained by the Tau method (Adeniyi, 1991; Adeniyi and Edungbola, 2008; Lanczos, 1938; Lanczos, 1956; Ojo and Adeniyi, 2012). Techniques based on this method have been studied with applications to more general equations including nonlinear ones (Ortiz, 1969; Yisa and Adeniyi, 2015).

In this direction, Lanczos (1975) developed a modification based on the used canonical polynomials and Ortiz (1969) proposed a recursive generalization of these polynomials to give some flexibility in the procedure involved. Adeniyi and Onumanyi (1991) reported a generalized Tau method based on the original formulation of the differential form and which also incorporated on error estimation of the Tau method with extension to segmented or piece-wise Tau approximation. Also, the three variants namely the differential, the integrated and the recursive formulations were discussed by Adeniyi and Aliyu (2008).

The first attempt on an error estimation of the Tau Method was by Lanczos in 1956 where he developed a simple algebraic approach by using the relation of the Chebyshev polynomials to trigonometric function, and which was applied only to the restricted class of first order problems. Fox (1968) later developed an approach which could handle higher order problems. However, Adeniyi (2008) constructed a polynomial error approximant of the error function $e_n(x)$ of the Lanczos Tau Method for ordinary differential equations, based on the error of Lanczos economization process, where he modified the approximant by perturbing some of the homogeneous conditions of $e_n(x)$. Yisa and Adeniyi (2012) developed a generalization of the canonical polynomials for over-determined m^{th} order ordinary differential equations (ODEs).

This paper is aimed at obtaining a better approximation by the differential form of Tau Method together with its error estimation for the problem (16.a)-(16.b), of which can be achieved by obtaining the Tau approximations of degree five to eight, the estimated error using

$\xi = \max_{0 \le x_k \le 1} |y_{n+1}(x_k) - y_n(x_k)|$ where $x_k = 0.01k$, for $k = 0(1)100$ and $5 \le n \le 8$. Finally, used to compare the accuracy of the two error estimation.

Differential Formulation of the Tau Method

Consider the m^{th} order linear differential equation with polynomial coefficients and polynomial right hand side:

$$Ly(x) := \sum_{r=0}^{N_r} (P_{rk} x^k) y^r(x) = \sum_{r=0}^{F} f_r x^r \quad (1a)$$

$$L^* y(x_{rk}) := \sum_{r=0}^{m-1} a_{rk} y^r(x_{rk}) = \alpha_k ; k = 0(1)m \quad (1b)$$

with smooth solution $y(x)$, $a \le x \le b, |a| < +\infty, |b| < +\infty$ satisfying (1a), where $x_{rk}, a_{rk}, \alpha_k, r = 0(1)m-1$ and $k = 1(1)m$ are real numbers, x_{rk} are point belonging to the interval $a \le x \le b$ for which the conditions (1b) are satisfied, P_{rk} and f_k are given constants and L is a differential operator.

The idea of Lanczos Tau Method is to approximate the solution of the differential system by an n^{th} degree polynomial function

$$y_n(x) = \sum_{r=0}^{n} a_r x^r \quad n \in \aleph \quad (2)$$

The perturbed equation is obtained by adding $H_n(x)$ to the right hand side of (1a), that is,

$$Ly_n(x) := \sum_{r=0}^{N_r} (P_{rk} x^k) y_n^r(x) = \sum_{r=0}^{F} f_r x^r + H_n(x) \quad (3a)$$

$$L^* y_n(x_{rk}) := \sum_{r=0}^{m-1} a_{rk} y_n^r(x_{rk}) = \alpha_k ; k = 0(1)m \quad (3b)$$

for $a \le x \le b$ and $\tau_r, r = 1(1)m+s$ are parameters to be determined together with the coefficients $a_r, r = 0(1)m+1$ in (2). The perturbation term $H_n(x)$ in (3a) is defined by

$$H_n(x) = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \quad (4)$$

$$T_n(x) = Cos\left\{ rCos^{-1}\left(\frac{2x - 2a}{b - a} - 1 \right) \right\} \equiv \sum_{r=0}^{n} C_r^{(n)} x^r \quad (5)$$

is the $r-th$ degree Chebyshev polynomial of first kind valid in the interval $a \le x \le b$ and

$$s = \max\{N_r - r : 0 \le r \le m\} \quad (6)$$

is the number of over determination of Equation (1a). To determine the coefficients $a_r, r = 0(1)n$ and $\tau_r, r = 1(1)m+s$ from the system of linear algebraic equation $A\underline{\tau} = \underline{b}$ obtained by equating the corresponding coefficients of powers of x in (3a)-(3b); where

$$A = (a_{ij}), 1 \le i, j \le n+m+s+1$$

$$\underline{b} = (b_i)^T, 1 \le i \le n+m+s+1$$

$$\underline{\tau} = (a_0, a_1, a_2, \cdots, a_n, \tau_1, \tau_2, \cdots, \tau_{m+s})^T$$

Consequently, we obtain from (2) our desired approximant $y_n(x)$ of $y(x)$

Error Estimation of the Differential Formulation of the Tau Method

Define the error function by

$$e_n(x) = y(x) - y_n(x) \quad (7)$$

Then $e_n(x)$ satisfies the perturbed error equation

$$Le_n(x) := Ly(x) - Ly_n(x) = -H_n(x) \quad (8)$$

together with the homogeneous conditions (3b) transform into $e_n(x_{rk}) = 0$.

By the method reported above 1.2, it is possible to construct an $(n+1)-th$ degree polynomial approximant (see Adeniyi and Onumanyi, 1991)

$$(e_n(x))_{n+1} = \max_{a \le x \le b} \left| \frac{\Phi_n \mu_m(x) T_{n-m+1}}{C_{n-m+1}^{(n-m+1)}} \right| \quad (9)$$

for a suitable approximation to $e_n(x)$ given by (7) where Φ_n is a constant to be determined and

$$\mu_m(x) = (x - x_0)^m \qquad a \le x \le b \quad (10)$$

The significance of $\mu_m(x)$ is to ensure that some or all the homogeneous conditions of $e_n(x)$ are satisfied by $(e_n(x))_{n+1}$; on the interval $a \le x \le b$, $C_{n-m+1}^{(n-m+1)}$ in (9) becomes $2^{2(n-m)+1}$ while (9) satisfies the perturbed problem

$$L(e_n(x))_{n+1} = -\sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) + \sum_{r=0}^{m+s-1} \tilde{\tau}_{m+s-r} T_{n-m+r+2}(x)$$

(11)

$$L^{*}(e_n(x_{rk}))_{n+1} = 0 \qquad k = 0(1)m \qquad (12)$$

where the extra parameters $\tilde{\tau}_r, r = 1(1)m+s$ are to be determined along with Φ_n.

Let ξ^{*} be defined in $a \le x \le b$, as

$$\xi^{*} = \max_{a\le x\le b} |(e_n(x))_{n+1}| + \max_{a\le x\le b} |e_n(x) - (e_n(x))_{n+1}| \qquad (13)$$

as n becomes very large (so that the second term on the RHS of (13) vanishes), Adeniyi and Onumanyi (1991) justifies the use of

$$\xi^{*} = \max_{a\le x\le b} |(e_n(x))_{n+1}| \qquad (14)$$

we insert (9) into (11) and then equating the corresponding coefficients of $x^{n+s+1}, x^{n+s}, \cdots, x^{n-m+1}$. The resulting linear system is solved for Φ_n by forward elimination and consequently, we obtain the approximate error estimate

$$\xi^{*} = \max_{a\le x\le b} |(e_n(x))_{n+1}| = \frac{|\Phi_n|}{|C^{(n-m+1)}_{n-m+1}|} \cong \max_{a\le x\le b} |e_n(x)| = \xi \qquad (15)$$

as an estimate error of the Tau method.

CLASS OF FOURTH ORDER DIFFERENTIAL EQUATION

Consider here the class of fourth order boundary value problems;

$$Ly(x) := y^{(iv)}(x) + \alpha_1 x^2 y'(x) + \alpha_2 x y(x) = 0 \quad a \le x \le b \qquad (16a)$$

$$y(a) = \rho_0,\ y'(a) = \rho_1,\ y'(b) = \lambda_0,\ y''(b) = \lambda_1 \qquad (16b)$$

where $\alpha_1, \alpha_2, \rho_0, \rho_1, \lambda_0$, and λ_1 are known real numbers. The differential variant of the Tau method will be adopted for this purpose. Without loss of generality, we shall assume the transformation from the interval $[a,b]$ to $[0,1]$ for convenience we use the transformation

$$u = \frac{x-a}{b-a};\ 0 \le x \le 1.$$

We shall consider here $m = 4, s = 1$, and substitute (2) and its derivatives into (16a) and then add (4) to have

$$Ly_n(x) := \sum_{r=0}^{n-4} r(r-1)(r-2)(r-3)a_r x^{r-4} + \sum_{r=0}^{n-1} \alpha_1 r a_r x^{r+1} + \sum_{r=0}^{n} \alpha_2 a_r x^{r+1} = \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \qquad (17)$$

$$Ly_n(x) := \sum_{r=0}^{n-4}(r+4)(r+3)(r+2)(r+1)a_{r+4}x^r + \sum_{r=0}^{n-1}\{\alpha_1(r-1)+\alpha_2\}a_{r-1}x^r + \{\alpha_1 n+\alpha_2\}a_n x^{n+1} + \alpha_2 a_{n-1}x^n$$
$$= \tau_1\sum_{r=0}^{n+1}C_r^{(n+1)}x^r + \tau_2\sum_{r=0}^{n}C_r^{(n)}x^r + \tau_3\sum_{r=0}^{n-1}C_r^{(n-1)}x^r + \tau_4\sum_{r=0}^{n-2}C_r^{(n-2)}x^r + \tau_5\sum_{r=0}^{n-3}C_r^{(n-3)}x^r$$

expanding above, we have

$$\sum_{r=0}^{n-4}\{(r+4)(r+3)(r+2)(r+1)a_{r+4} + [\alpha_1(r-1)+\alpha_2]a_{r-1}\}x^r + \{\alpha_1 n+\alpha_2\}a_n x^{n+1} + \alpha_2 a_{n-1}x^n +$$
$$\{\alpha_1(n-2)+\alpha_2\}a_{n-2}x^{n-1} + \{\alpha_1(n-3)+\alpha_2\}a_{n-3}x^{n-2} + \{\alpha_1(n-4)+\alpha_2\}a_{n-4}x^{n-3} =$$
$$\sum_{r=0}^{n-4}\{\tau_1 C_r^{(n+1)} + \tau_2 C_r^{(n)} + \tau_3 C_r^{(n-1)} + \tau_4 C_r^{(n-2)} + \tau_5 C_r^{(n-3)}\}x^r + \tau_1 C_{n+1}^{(n+1)}x^{n+1} + \{\tau_1 C_n^{(n+1)} + \tau_2 C_n^{(n)}\}x^n +$$
$$\{\tau_1 C_{n-1}^{(n+1)} + \tau_2 C_{n-1}^{(n)} + \tau_3 C_{n-1}^{(n-1)}\}x^{n-1} + \{\tau_1 C_{n-2}^{(n+1)} + \tau_2 C_{n-2}^{(n)} + \tau_3 C_{n-2}^{(n-1)} + \tau_4 C_{n-2}^{(n-2)}\}x^{n-2} +$$
$$\{\tau_1 C_{n-3}^{(n+1)} + \tau_2 C_{n-3}^{(n)} + \tau_3 C_{n-3}^{(n-1)} + \tau_4 C_{n-3}^{(n-2)} + \tau_5 C_{n-3}^{(n-3)}\}x^{n-3}$$

Simplifying above and equating the corresponding coefficients of powers of $x^{n+1}, x^n, \cdots, x^{n-3}$ to zero to get the recurrence relations of the form

$$(r+4)(r+3)(r+2)(r+1)a_{r+4} + \{\alpha_1(r-1)+\alpha_2\}a_{r-1} - \tau_1 C_r^{(n+1)} - \tau_2 C_r^{(n)} - \tau_3 C_r^{(n-1)} - \tau_4 C_r^{(n-2)} - \tau_5 C_r^{(n-3)} = 0$$
$$\{\alpha_1 n+\alpha_2\}a_n - \tau_1 C_{n+1}^{(n+1)} = 0 \quad r = 0(1)n-4$$

$$\alpha_2 a_{n-1} - \tau_1 C_n^{(n+1)} - \tau_2 C_n^{(n)} = 0$$
$$\{\alpha_1(n-2)+\alpha_2\}a_{n-2} - \tau_1 C_{n-1}^{(n+1)} - \tau_2 C_{n-1}^{(n)} - \tau_3 C_{n-1}^{(n-1)} = 0$$
$$\{\alpha_1(n-3)+\alpha_2\}a_{n-3} - \tau_1 C_{n-2}^{(n+1)} - \tau_2 C_{n-2}^{(n)} - \tau_3 C_{n-2}^{(n-1)} - \tau_4 C_{n-2}^{(n-2)} = 0$$
$$\{\alpha_1(n-4)+\alpha_2\}a_{n-4} - \tau_1 C_{n-3}^{(n+1)} - \tau_2 C_{n-3}^{(n)} - \tau_3 C_{n-3}^{(n-1)} - \tau_4 C_{n-3}^{(n-2)} - \tau_5 C_{n-3}^{(n-3)} = 0$$

We solve this system together with the four equations arising from the conditions (16b), so as to determine $a_r, r = 0(1)n$ in (16a). Consequently we obtain the desired approximant of $Y(x)$.

ERROR ESTIMATION FOR A CLASS OF FOURTH ORDER DIFFERENTIAL EQUATION

For the problem (16), and the perturbation term (11), we have

$$L(e_n(x))_{n+1} := \left(\frac{d^4}{dx^4} + \alpha_1 x^2 \frac{d}{dx} + \alpha_2 x\right)(e_n(x))_{n+1} = \sum_{r=0}^{m+s-1} \tilde{\tau}_{m+s-r} T_{n-m+r+2}(x) - \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x)$$

(18a)

which satisfies the homogeneous conditions

$$(e_n(0))_{n+1} = (e'_n(0))_{n+1} = (e'_n(1))_{n+1} = (e'_n(1))_{n+1} = 0 \qquad (18b)$$

where $(e_n(x))_{n+1}$ is defined in (9).

We shall consider two different cases of $\mu_m(x)$

Case I: $\mu_m(x) = (x - x_0)^m = x^m$ where $0 \le x \le 1$

For $m = 4, s = 1,$ we have

$$(e_n(x))_{n+1} = \frac{\Phi_n \sum_{r=0}^{n-3} C_r^{(n-3)} x^{r+4}}{C_{n-3}^{(n-3)}} \qquad (19)$$

$$(e_n(x))_{n+1} = \Theta \sum_{r=0}^{n-3} C_r^{(n-3)} x^{r+4} \qquad (20)$$

where $\Theta = \Phi_n \left(C_{n-3}^{(n-3)}\right)^{-1}$ (21)

Substituting (20) into (18) and simplifying to obtain

$$L(e_n(x))_{n+1} := \Theta \sum_{r=0}^{n-3} C_r^{(n-3)} \{((r+4)\alpha_1 + \alpha_2) x^{r+5} + (r+4)(r+3)(r+2)(r+1) x^r\} = \tau_1 \sum_{r=0}^{n+2} C_r^{(n+2)} x^r +$$
$$(\tilde{\tau}_2 - \tau_1) \sum_{r=0}^{n+1} C_r^{(n+1)} x^r + (\tilde{\tau}_3 - \tau_2) \sum_{r=0}^{n} C_r^{(n)} x^r + (\tilde{\tau}_4 - \tau_3) \sum_{r=0}^{n-1} C_r^{(n-1)} x^r + (\tilde{\tau}_5 - \tau_4) \sum_{r=0}^{n-2} C_r^{(n-2)} x^r - \tau_5 \sum_{r=0}^{n-3} C_r^{(n-3)} x^r + \cdots$$

Expanding the above equation and equating the corresponding coefficients of powers of $x^{n+2}, x^{n+1}, \cdots, x^{n-3}$, to obtain the system of equations

$$\tilde{\tau}_1 C_{n+2}^{(n+2)} = \Theta((n+1)\alpha_1 + \alpha_2) C_{n-4}^{(n-3)}$$
$$\tilde{\tau}_1 C_{n+1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n+1}^{(n+1)} = \Theta(n\alpha_1 + \alpha_2) C_{n-4}^{(n-3)}$$
$$\tilde{\tau}_1 C_{n}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n}^{(n)} = \Theta((n-)\alpha_1 + \alpha_2) C_{n-5}^{(n-3)}$$
$$\tilde{\tau}_1 C_{n-1}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-1}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-1}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-1}^{(n-1)} = \Theta((n-2)\alpha_1 + \alpha_2) C_{n-6}^{(n-3)}$$
$$\tilde{\tau}_1 C_{n-2}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-2}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-2}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-2}^{(n-1)} + (\tilde{\tau}_5 - \tau_4) C_{n-2}^{(n-1)} = \Theta((n-3)\alpha_1 + \alpha_2) C_{n-7}^{(n-3)}$$
$$\tilde{\tau}_1 C_{n-3}^{(n+2)} + (\tilde{\tau}_2 - \tau_1) C_{n-3}^{(n+1)} + (\tilde{\tau}_3 - \tau_2) C_{n-3}^{(n)} + (\tilde{\tau}_4 - \tau_3) C_{n-3}^{(n-1)} + (\tilde{\tau}_5 - \tau_4) C_{n-3}^{(n-1)} - \tau_5 C_{n-3}^{(n-3)} = \Theta\{((n-4)\alpha_1 + \alpha_2) C_{n-8}^{(n-3)}$$
$$+ (n^4 - 2n^3 - n^2 + 2n)\} C_{n-3}^{(n-3)}$$

we solve the above equations for $\hat{\tau}_r$ by forward elimination method using the relation

$$C_n^{(n)} = 2^{2n-1}; C_{n-1}^{(n)} = -\frac{1}{2} n C_n^{(n)} = -n 2^{2n-2}$$

to obtain

$$\Theta = \frac{2^{4n-4} \tau_5}{K} \qquad (22)$$

Equating (21) and (22), we have

$$\Phi_n = \frac{2^{6n-11} \tau_5}{K} \qquad (23)$$

$$(e_n(x))_{n+1} = \frac{\Phi_n}{C_{n-3}^{(n-3)}} = \frac{2^{4n-4} \tau_5}{K} \qquad (24)$$

Thus, taking the maximum of (24) yield the estimate

$$\xi_1^* = \max_{0 \le x \le 1} |(e_n(x))_{n+1}| = \frac{|\Phi_n|}{|C_{n-3}^{(n-3)}|} = \frac{2^{4n-4} |\tau_5|}{|K|} \qquad (25)$$

Where

$$K = \{C_{n-3}^{(n+2)} B + C_{n-3}^{(n+1)} A + C_{n-3}^{(n)} \Delta + C_{n-3}^{(n-1)} X - ((n-4)\alpha_1 + \alpha_2) C_{n-8}^{(n-3)}\} 2^{3+2n}$$
$$- \{2(n-2)E + (n^4 - 2n^3 - n^2 + 2n)\} 2^{4(n-1)}$$
$$B = ((n+1)\alpha_1 + \alpha_2) 2^{-10}$$
$$A = 2(n+2)B - (n-2)(n\alpha_1 + \alpha_2)$$
$$\Delta = \{((n-1)\alpha_1 + \alpha_2) C_{n-5}^{(n-3)} - C_n^{(n+2)} B\} 2^{1-2n} + 2(n+1)A$$
$$X = \{((n-2)\alpha_1 + \alpha_2) C_{n-6}^{(n-3)} - C_{n-1}^{(n+2)} B - C_{n-1}^{(n+1)} A\} 2^{3-2n} + 2n\Delta$$
$$E = \{((n-3)\alpha_1 + \alpha_2) C_{n-7}^{(n-3)} - C_{n-2}^{(n+2)} B - C_{n-2}^{(n+1)} A - C_{n-2}^{(n)} \Delta\} 2^{5-2n} + 2(n-1)X$$

Case II: $\mu_4(x) = x^2(x-1)^2$

Thus, (19) becomes

$$(e_n(x))_{n+1} = \Theta \sum_{r=0}^{n-3} C_r^{(n-3)} \left(x^{r+4} - 2x^{r+3} + x^{r+2}\right) \qquad (26)$$

Following the same procedures as **case I**, we obtain

Table 1. Error estimation and estimated error for Problem 1.

Degree n	Error Estimation ξ^*		Estimated Error Using ξ
	$\mu_4(x) = x^4$	$\mu_4(x) = x^2(x-1)^2$	
5	6.5525×10^{-1}	7.1791×10^{-1}	7.3867×10^{-1}
6	2.3089×10^{-4}	2.3091×10^{-4}	2.3092×10^{-4}
7	1.8899×10^{-4}	1.9627×10^{-4}	1.9738×10^{-4}

Remark: The error estimate accurately captures the order of the estimated error.

Table 2. Error estimation and estimated error for Problem 2.

Degree n	Error Estimation ξ^*		Estimated Error Using ξ
	$\mu_4(x) = x^4$	$\mu_4(x) = x^2(x-1)^2$	
5	1.8156×10^{-3}	3.6519×10^{-4}	3.6796×10^{-4}
6	5.6649×10^{-4}	5.0402×10^{-4}	4.7024×10^{-5}
7	4.2202×10^{-4}	5.1888×10^{-4}	9.0000×10^{-4}

Remark: The higher order approximation show closeness to the estimated error for both the choices of $\mu_m(x)$. The under estimation of the error results from low degree approximation.

$$(e_n(x))_{n+1} = \frac{\Phi_n}{C_{n-3}^{(n-3)}} = \frac{2^{4n-4}\tau_5}{\mathrm{N}} \quad (27)$$

Thus, taking the maximum of (27) yield the estimate

$$\xi_2^* = \max_{0\le x\le 1} |(e_n(x))_{n+1}| = \frac{|\Phi_n|}{|C_{n-3}^{(n-3)}|} = \frac{2^{4n-4}|\tau_5|}{|\mathrm{N}|} \quad (28)$$

Where

$$\mathrm{N}=\left\{C_{n-3}^{(n+2)}\beta+C_{n-3}^{(n+1)}\chi+C_{n-3}^{(n)}\delta+C_{n-3}^{(n-1)}\gamma-((n-4)\alpha_1+\alpha_2)\left(C_{n-6}^{(n-3)}-2C_{n-7}^{(n-3)}+C_{n-8}^{(n-3)}\right)\right\}2^{3+2n}$$
$$-\left\{2(n-2)\eta+\left(n^4-2n^3-n^2+2n\right)\right\}2^{4(n-1)}$$
$$\beta=((n+1)\alpha_1+\alpha_2)2^{-10}$$
$$\chi=2(n+2)\beta-(n+1)(n\alpha_1+\alpha_2)2^{-9}$$
$$\delta=((n-1)\alpha_1+\alpha_2)(n-2)2^{-6}+\left\{((n-1)\alpha_1+\alpha_2)C_{n-5}^{(n-3)}-C_{n}^{(n+2)}\beta\right\}2^{1-2n}+2(n+1)\chi$$
$$\gamma=\left\{((n-2)\alpha_1+\alpha_2)\left(C_{n-6}^{(n-3)}-2C_{n-5}^{(n-3)}\right)-C_{n-1}^{(n+2)}\beta-C_{n-1}^{(n-1)}\chi\right\}2^{3-2n}-((n-2)\alpha_1+\alpha_2)(n-3)2^{-5}+2n\gamma$$
$$\eta=\left\{((n-3)\alpha_1+\alpha_2)\left(C_{n-5}^{(n-3)}-2C_{n-6}^{(n-3)}+C_{n-7}^{(n-3)}\right)-C_{n-2}^{(n+2)}\beta-C_{n-2}^{(n+1)}\chi-C_{n-2}^{(n)}\delta\right\}2^{5-2n}+2(n-1)\gamma$$

NUMERICAL EXPERIMENT

In this section, four selected problems were considered for experiments. To solve these differential equations using the method developed in the previous sections and the results will be presented in tabular form, the estimated error obtained as $\xi = \max_{0\le x_k \le 1} |y_{n+1}(x_k) - y_n(x_k)|$ where $x_k = 0.01k$, for $k = 0(1)100$ and $5 \le n \le 8$.

Problem 1

$$y^{(iv)}(x) + x^2 y'(x) + xy(x) = 0 \quad 0 \le x \le 1 \quad (29a)$$

$$y(0) = 1,\ y'(0) = -1,\ y'(1) = 0,\ y''(1) = 0.5 \quad (29b)$$

Problem 2

$$y^{(iv)}(x) - \frac{1}{2}x^2 y'(x) + \frac{3}{2}xy(x) = 0 \quad 0 \le x \le 1 \quad (30a)$$

$$y(0) = Cos(0),\ y(0) = 1,\ y'(1) = Sinh(1),\ y''(1) = Cosh(1) \quad (30b)$$

Problem 3

$$y^{(iv)}(x) - 6x^2 y'(x) = 0 \quad 0 \le x \le 1 \quad (31a)$$

$$y(0) = 1,\ y'(0) = -\frac{1}{2},\ y'(1) = Sinh(1),\ y''(1) = 0 \quad (31b)$$

Table 3. Error estimation and estimated error for Problem 3.

Degree n	Error Estimation ξ^*		Estimated Error Using ξ
	$\mu_4(x) = x^4$	$\mu_4(x) = x^2(x-1)^2$	
5	2.2653×10^{-1}	2.4496×10^{-1}	2.4520×10^{-1}
6	4.3494×10^{-3}	4.4596×10^{-3}	4.4699×10^{-3}
7	8.3538×10^{-3}	8.9012×10^{-3}	9.1083×10^{-3}

Remark: Error and error estimate compare favourably well for the two forms of $\mu_m(x)$.

Table 4. Error estimation and estimated error for Problem 4.

Degree n	Error Estimation ξ^*		Estimated Error Using ξ
	$\mu_4(x) = x^4$	$\mu_4(x) = x^2(x-1)^2$	
5	1.7685×10^{-3}	1.7832×10^{-2}	1.7842×10^{-2}
6	3.3567×10^{-4}	3.4089×10^{-4}	3.4242×10^{-4}
7	9.5154×10^{-4}	9.3672×10^{-4}	9.4831×10^{-5}

Remark: The higher order approximation show closeness to the estimated error for both choices of $\mu_m(x)$. The under estimation of the error results from the low degree approximation.

Problem 4

$$y^{(iv)}(x) - 6x^2y'(x) + 4xy(x) = 0 \quad 0 \le x \le 1 \qquad (32a)$$

$$y(0) = 1,\ y(0) = Cosh\ (0),\ y'(1) = Sinh\ (1),\ y''(1) = 1.75 \qquad (32b)$$

Conclusion

The derivation of a general formula of the differential variant with its error estimation of the Tau method for a certain class of fourth order Boundary Value Problems (BVPs) with first degree over-determination has been presented. The recursive formulae will give easy determination of particular cases of nth degree approximation specified. It was reported that, both the two choices of $\mu_m(x)$ captures the order of the estimated error for higher degree approximation as shown in Tables 1 and 2, and the other two conditions $e'(1) = e''(1) = 0$ are perturbed for the second choices of $\mu_m(x)$ that is $\mu_4(x) = x^2(x-1)^2$ which gives suitable approximation. The under estimation of the error by our error estimate is mainly due to the low degree approximation.

REFERENCES

Adeniy RB, Aliyu IM (2008). On Tau method for the class of non-overdetermined second order differential equations. J. Niger. Assoc. Math. Phys. 12:387-398.

Adeniyi RB (1991). On the Tau method for numerical solution of ordinary differential equations, Doctoral Theses (Unpublished), University of Ilorin, Nigeria.

Adeniyi RB (2008). An improved error estimation of the Tau method for boundary value problems in ordinary differential equations. Res. J. Appl. Sci. 3(6):456-464.

Adeniyi RB, Edungbola EO (2008). On the Tau method for certain over-determined first order differential equations. J. Niger. Assoc. Math. Phys. 12:399-408.

Adeniyi RB, Onumanyi P (1991). Error estimation in the numerical solution of ordinary differential equations with the Tau method. Comp. Maths. Appl. 21(9):19-27.

Fox L (1968). Numerical solution of ordinary and partial differential equations, Pergmen Press, Oxford

Lanczos C (1938). Trigonometric interpolation of empirical and analytic functions. J. Maths. Phys. 17:123-199.

Lanczos C (1956). Applied analysis, Englewood Cliffs, Prentice Hall, New Jersey.

Lanczos C (1975). Step by step Tau Method-Part I. Piecewise polynomial approximations, Imperial College, University of London, London, English.

Ojo VO, Adeniyi RB (2012). The differential form of the Tau method and its error estimate for third order non-overdetermined differential equations. Gen. Maths. Note 11(1):41-49.

Ortiz EL (1969). The Tau Method, SAIM J. Numer. Anal. 6:480-492.

Yisa BM, Adeniyi RB (2012). On the variants of the Tau method for solution of IVPs in the first order differential system. Kenya J. Sci.15:1-12.

Yisa BM, Adeniyi RB (2015). On generalization of the error and error estimation process of Ortiz's recursive formulation of the Tau method. J. Math. Soc. 34:70-82.