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Full Length Research Paper

A shorter solution to the Clay millennium problem about regularity of the Navier-Stokes equations

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The Clay millennium problem regarding the Navier-Stokes equations is one of the seven famous mathematical problems for which the Clay Mathematics Institute has set a high monetary award for its solution. It is considered a difficult problem because it has refused to solve it for almost a whole century. The Navier-Stokes equations, which are the equations that govern the flow of fluids, were formulated long ago in mathematical physics, before matter was known to be composed of atoms. So in effect they formulated the old infinitely divisible material fluids. Although it is known that the set of Navier-Stokes equations has a unique smooth local time solution under the assumptions of the millennium problem, it is not known whether this solution can always be smooth and globally extended, called the regularity of the Navier-Stokes equations in 3 dimensions. We are concerned of course with solutions of the Navier-Stokes equations as in the initial Schwartz data in Fefferman (2006) that are smooth at least in a small time interval [0,t) otherwise the well-known proof of uniqueness of the solutions for the Navier-Stokes equations would not hold and the millennium problem would be considered ill-posed . The corresponding case of regularity in 2 dimensions has long ago been shown to hold, but the 3-dimensionality refuses to prove it. Of course, the natural outcome would be that the regularity also holds for 3 dimensions. Many people feel that this difficulty hides our lack of understanding of the 3-dimensional flow laws of incompressible fluids. Compared to the older solution proposed by Kyritsis (2021a, 2013), this paper presents a shorter solution to the Clay Millennium problem about the Navier-Stokes equations. The longer solution is based on the equivalence of smooth Schwartz initial data in the original formulation of the problem with simply connected compact and smooth boundary initial data (e.g., on a 3-ball, see Kyritsis, 2017a). The current short solution is in the context of smooth Schwartz initial data and is an independent solution logically different from the previous one. The next strategy is as follows: (1) from the finite initial energy and energy conservation, and due to the incompressibility as well as the conservative field of the pressure forces, we obtain the regularity in the pressures; (2) from the regularity in the pressures, we obtain the regularity of the material velocities, which leads to the regularity of the Navier-Stokes equations.

Key words: Incompressible flows, regularity, Navier-Stokes equations, 4^{th} Clay millennium problem.

Mathematical Subject Classification: 76A02

INTRODUCTION

The Clay millennium problem about the regularity of the Navier-Stokes equations problem was solved during the spring of 2013 (later uploaded; Kyritsis, 2013) and was also later uploaded as a preprint in February 2018 (Kyritsis, 2018; Kyritsis 2019), and then published again in August by Kyritsis (2021a). The solution has also been published as a chapter in a book by Kyritsis (2021c) and also as a whole book devoted to it by Kyritsis (2021d). The latter book also contains the solution of the 3^{rd} Clay Millennium problem in computational complexity, since the author has solved 2 of the 7 millennium problems (Kyritsis, 2021b). In the paper by Kyritsis (2017b), there is also an alternative 3rd solution to the 4th Clay Millennium problem based on the additional hypothesis of the conservation of particles. The other 2 solutions have no additional hypothesis other than those of the formulation of the problem by the Clay Mathematical Institute.

This millennium problem was solved not only by Kyritsis (2021a) and Kyritsis (2013), but also by other authors, such as Durmagambetov and Fazilova (2015) and Moschandreou (2021).

An attempt has been made to keep the length of this paper as short as possible to encourage reading it and to make the solution as easy to understand as possible. Therefore, this study shall omit its formal formulation, which can be found in Fefferman (2006) or Kyritsis (2021a). In addition, this study shall omit peripheral results, such as new necessary and sufficient conditions of regularity as in Kyritsis (2021a) that are not directly applied in the solutions. This paper includes only the absolutely necessary line of arguments in order for it to be an independent logically different and shorter solution than the previous one by the same author. Within the context of smooth Schwartz initial data, which is one of the initial formulations of the problem, the application of Kyritsis (2017a) is avoided. The latter paper is a slight modification of the proof of Theorem 12.2 by Tao (2013), who proved that the regularity of Navier-Stokes on a 3D torus (periodic formulation) implies regularity on smooth Schwartz initial data; therefore, almost the same proof also proves that the regularity of Navier-Stokes on a 3-ball smooth compactly supported initial data implies regularity on smooth Schwartz initial data. We must add one important point here. In order for the formal formulation that can be found in (Fefferman 2006) not to be ill-posed, we must add that **we are concerned with solutions of the Navier-Stokes** equations with initial Schwartz data as in (Fefferman 2006), **that are smooth at least within a small time interval [0,t)**. This is a necessary hypothesis to add the initial data hypothesises by Fefferman (2006); otherwise, the standard proof of the **uniqueness of solutions of the Navier-Stokes equations** mentioned in the formal formulation, which proof can be found in (Majda and Bertozzi, 2002) would not hold!

THE STRATEGY

The next strategy is as follows:

1) From the finite initial energy and energy conservation, and due to the incompressibility as well as the conservative field of the pressure forces, we obtain the regularity (uniform-in-time boundedness) in the pressures.
2) From the regularity in the pressures, we obtain the regularity (uniform-in-time boundedness) of the material velocities, which leads to the final regularity of the Navier-Stokes equations.

SOME KNOWN OR DIRECTLY DERIVABLE USEFUL RESULTS THAT WILL BE USED

In this paragraph, some known theorems and results will be stated, which will be used in this paper, or for the convenience of the reader to know, so that the reader can understand them at a glance without searching in the literature - image of what already exists and what has been proved.

Remark 3.1: Finite initial energy and energy conservation equations

When we want to prove that the smoothness in the local time solutions of the Euler or Navier-Stokes equations is conserved and that they can be extended infinitely in time, we usually apply a “reductio ad absurdum” argument: let the maximum finite time T* and the interval [0,T*) be such that the local solution can be extended smoothly in it. Then the time T* will be a blow-up time, if we manage to smoothly extend the solutions on the interval [0,T*]. Then there is no finite Blow-up time T* and the solutions holds in [0,+∞). The necessary and sufficient conditions for this extension to be possible are listed below. Obviously, no smoothness assumption can be made for time T*, as this is what must be proved. However, we can still assume that at T*, the energy conservation and momentum conservation will hold even for a singularity at T*, since these are universal laws of nature and calculating their integrals does not require smooth functions, but only integrable functions, possibly with points of discontinuity.

A very well-known form of the energy conservation equation and accumulative energy dissipation is given as follows:

$$\frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx + v\int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt = \frac{1}{2}\int_{R^3}\|u(x,0\|^2 dx \quad (1)$$

Where $E(0) = \frac{1}{2}\int_{R^3}\|u(x,0)\|^2 dx$ (2)

is the initial finite energy,

$$E(T) = \frac{1}{2}\int_{R^3}\|u(x,T)\|^2 dx \quad (3)$$

is the final finite energy

and $\Delta E = v\int_0^T\int_{R^3}\|\nabla u(x,t)\|^2 dxdt$ (4)

is the accumulative finite energy dissipation from time 0 to time T. This is due to the viscosity of the internal heat of the fluid. For the Euler equation, it is zero. Obviously,

$\Delta E <= E(0) >= E(T)$ (5)

The rate of energy dissipation is given by

$$\frac{dE}{dt}(t) = -\nu \int_{R^3} \|\nabla u\|^2 dx < 0 \quad (6)$$

where v is the density-normalized viscosity coefficient (Majda and Bertozzi, 2002; Proposition 1.13, Equation 1.80, pp. 28).

Remark 3.2: Next are 3 very useful inequalities for the unique local time [0,T] with smooth solutions u of the Euler and Navier-Stokes equations having smooth Schwartz initial data and finite initial energy (they hold for more general initial data of conditions, but we will not use it):

We use $||.||_m$ to denote the Sobolev norm of order m. Thus, if m = 0, this is essentially the L_2-norm. We use $||.||_{L\infty}$ to denote the supremum norm, where u is the velocity, ω is the vorticity, and c_m, c are constants.

1) $$\|u(x,T)\|_m \leq \|u(x,0)\|_m \exp(\int_0^T c_m \|\nabla(u(x,t)\|_{L_\infty} dt) \quad (7)$$

(Majda and Bertozzi, 2002; proof of Theorem 3.6, pp. 117, Equation 3.79).

2) $$\|\omega(x,t)\|_0 \leq \|\omega(x,0)\|_0 \exp(c\int_0^T \|\nabla u(x,t)\|_{L_\infty} dt) \quad (8)$$

(Majda and Bertozzi, 2002; proof of Theorem 3.6, pp. 117, Equation 3.80).

3) $$\|\nabla u(x,t)\|_{L_\infty} \leq \|\nabla u(x,0)\|_0 \exp(\int_0^t \|\omega(x,s)\|_{L_\infty} ds) \quad (9)$$

(Majda and Bertozzi, 2002; proof of Theorem 3.6, pp. 118, last equation of the proof).

Next is the well-known list of necessary and sufficient conditions for the regularity (global time existence and smoothness) of the solutions to the Euler and Navier-Stokes equations, under the standard assumptions of the Clay Millennium problem with smooth Schwartz initial data. We denote by T* the maximum Blow-up time (if it exists) at which the local solution u(x,t) is smooth in [0,T*).

DEFINITION 3.2

When we write that ***a quantity Q(t) of the flow, which generally depends on time, is uniform in time bounded during the flow****, we mean that there is a bound M independent of time, such that Q(t)<=M for all t in [0, T*).*

It is important to remark here that the existence and uniqueness are local in time (well-posedness) and apply not only to the case of viscous flows following the Navier-Stokes equations, but also to the case of inviscid flows under Euler's equations. There are many other papers and authors that have proven the local existence and uniqueness of smooth solutions both for the Navier-Stokes and Euler equation with the same methodology, where the value of the viscosity coefficient v = 0 can as well be included (Majda and Bertozzi, 2002; p. 104, Theorem 3.4, paragraph 3.2.3 and paragraph 4.1, p. 138).

PROPOSITION 3.1 (Local well-posedness in H^1). *Let (u_0, f, T) be H^1 data*

(i) Strong solution: If (u, p, u_0, f, T) is an H^1 mild solution, then

$$u \in C_t^0 H_x^1([0,T] \times R^3)$$

(ii) Local existence and regularity: If

$$(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})^4 T < c$$

for a sufficiently small absolute constant c > 0, then there exists an H^1 mild solution (u, p, u_0, f, T) with the indicated data, where

$$\|u\|_{X^k([0,T] \times R^3)} = O(\|u_0\|_{H_X^1(R^3)} + \|f\|_{L_t^1 H_X^1(R^3)})$$

and more generally

$$\|u\|_{X^k([0,T] \times R^3)} = O_k(\|u_0\|_{H_X^k(R^3)}, \|f\|_{L_t^1 H_X^1(R^3)}, 1)$$

for each k>=1. In particular, one has local existence whenever T is sufficiently small, depending on the norm H^1(u_0, f, T).

(iii) Uniqueness: There is at most one H^1 mild solution (u, p, u_0, f, T) with the indicated data.

(iv) Regularity: If (u, p, u_0, f, T) is an H^1 mild solution and (u_0, f, T) is (smooth) Schwartz data, then u and p are smooth solutions; in fact, one has

$\partial_t^j u, \partial_t^j p \in L_t^\infty H^k([0,T] \times R^3)$ *for all j, K >=0.*

For the proof of the above theorem, the reader is referred to (Tao, 2013; theorem 5.4), but also to the papers and books of the other authors mentioned above.

PROPOSITION 3.2 (Necessary and sufficient condition for regularity)

The local solution u(x,t), t in [0,T) of the Euler or Navier-*

Stokes equations has smooth Schwartz initial data that can be extended to [0,T], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the* ***Sobolev norm ||u(x,t)||m****, m>=3/2+2, remains bounded, and the same bound exists in all of [0,T*), then, there is no maximal Blow-up time T*, and the solution exists smooth in [0,+∞)*

Remark 3.3: See for a proof (Majda and Bertozzi, 2002, p. 115, line 10 from below)

PROPOSITION 3.3 (Necessary and sufficient condition for regularity; Beale et al., 1984)

The local solution u(x,t), t in [0,T) of the Euler or Navier-Stokes equations has smooth compact support initial data that can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if in the finite time interval [0,T*] there exist a bound M>0 such that the* ***vorticity is bounded by M that accumulates*** *in [0,T*]:*

$$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty}\, dt \le M \qquad (10)$$

Then there is no maximal Blow-up time T and the solution exists smoothly in [0,+∞)*

Remark 3.4: (Majda and Bertozzi, 2002, p. 115, Theorem 3.6). For the case of inviscid flows, there is also Theorem 5.1 on p. 171 (Lemarie-Rieusset, 2002). Conversely, if regularity holds, then the vorticity is supremum bounded in any interval of the smoothness in a compact connected set. The above theorems in the book by Majda and Bertozzi (2002) guarantee that the above conditions extend the local time solution to the global time; that is, to solutions (u, p, u_0, f, T) of the H^1 mild solution, **for any T**. Then applying part (iv) of PROPOSITION 3.1 above, we obtain that this solution is also smooth in the classical sense, for all T>0, and thus is globally smooth in time.

PROPOSITION 3.4 (Necessary and sufficient condition of vorticity for regularity)

The local solution u(x,t), t in [0,T) of the Euler or Navier-Stokes equations has smooth compact support initial data that can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0 such that the* ***vorticity is bounded by M in the supremum norm*** *L∞ in [0,T*] and on any compact set:*

$$\|\omega(x,t)\|_{L_\infty} \le M \quad \text{for all t in [0,T*)} \qquad (11)$$

Then there is no maximal Blow-up time T and the solution exists smoothly in [0,+∞)*

Remark 3.5: Obviously if $\|\omega(x,t)\|_{L_\infty} \le M$, then the integral also exists and is bounded:

$\int_0^{T^*} \|\omega(x,t)\|_{L_\infty}\, dt \le M_1$ and the previous Proposition 3.3 applies. Conversely, if regularity holds, then the vorticity is supremum bounded in any interval of the smoothness in a compact connected set.

PROPOSITION 3.5 (Necessary and sufficient condition for regularity)

The local solution u(x,t), t in [0,T) of the Euler or Navier-Stokes equations has smooth Schwartz initial data that can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if for the finite time interval [0,T*], there exist a bound M>0 such that the space partial derivatives or Jacobean is bounded by M in the supremum norm L∞ in [0,T*]:*

$$\|\nabla u(x,t)\|_{L_\infty} \le M \quad \text{for all t in [0,T*)} \qquad (12)$$

Then there is no maximal Blow-up time T and the solution exists smoothly in [0,+∞)*

Remark 3.6: It comes directly from the inequality (Equation 7) and the application of Proposition 3.2. Conversely, if regularity holds, then the space derivatives are supremum bounded in any finite time interval of smoothness.

PROPOSITION 3.6 (Fefferman, 2006; Necessary and sufficient condition of velocities for regularity)

The local solution u(x,t), t in [0,T) of the Euler or Navier-Stokes equations has smooth Schwartz initial data that can be extended to [0,T*], where T* is the maximal time that the local solution u(x,t) is smooth in [0,T*), if and only if the velocities ||u(x,t)|| do not become unbounded as t->T*. Then there is no maximal Blow-up time T* and the solution exists smoothly in [0,+∞).*

Remark 3.7: This is mentioned in the standard formulation of the Clay Millennium problem (Fefferman, 2006, p. 2, line 1 from below): quote "...For the Navier-Stokes equations (v>0), if there is a solution with a finite blow-up time T, then the velocities $u_i(x,t)$, 1<=i<=3 become unbounded near the blow-up time." The converse-negation of this is that if the velocities remain bounded near T*, then there is no blow-up at T* and the solution is regular or globally smooth in time. Conversely, of course, if regularity holds, then the velocities, due to smoothness, are supremum bounded in the compact set during any finite time interval.

The study did not find such a theorem specifically in the books or papers being studied, but took this condition for granted along with the formulation of the problem.

A probable line of arguments to prove it might be as follows:

We want to prove that a blow-up cannot occur only in the spatial partial derivatives of the velocities, but not in the velocities themselves. If such a strange blow-up occurs, then, as in Proposition 3.5, the Jacobean of the velocities will blow up. This provides that the convective acceleration

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \bullet \nabla u$$

also blows up (since the term in the convective acceleration of the partial derivative of the velocity remains bounded by the hypothesis of not blowing up velocities or oscillating wildly).

$\frac{Du}{Dt}$->+∞ will blow up as t->T*,

Thus, by integrating on a path trajectory $\int_0^T \frac{Du}{Dt} dt$, we deduce that the velocity on the trajectory blows up, which contradicts the initial hypothesis. Therefore, the flow in [0,T*] is regular, as claimed by Fefferman (2006) in Proposition 3.6.

We notice that this condition of Fefferman applies only to viscous flows, but since Proposition 3.4 holds for inviscid flows under the Euler equations, this necessary and sufficient condition also holds for inviscid flows.

SUFFICIENT CONDITIONS FOR THE REGULARITY OF PRESSURE

PROPOSITION 4.1 (Pressure, a sufficient condition for regularity)

Let the local solution u(x,t), t in [0,T) of the Navier-Stokes equations with non-zero viscosity have smooth Schwartz initial data, then it can be extended to [0,T*], where T* is the maximal time at which the local solution u(x,t) is smooth in [0,T*) (we include the trivial case T^* =+∞), and thus can be extended to all times [0,+∞); in other words, the solution is regular if there exists a time uniform bound M for the pressure p. In other words, such that p≤ M for all t in [0,T*). Still in other words, smoothness and boundedness of the pressure p on the 3-space and in finite time interval [0,T] are characteristic conditions for regularity.*

Proof: Let us derive the regularity starting from the smoothness and boundedness of this characteristic of the pressure p in 3-space and in finite time interval [0,T].

We notice that in the Navier-Stokes equation of incompressible fluids, the pressure forces define a ***conservative force-field***, since it is the gradient of a scalar-field of pressure p, which plays the role of a scalar potential. Moreover, this property as a **conservative force-field** is **an invariant** in the flow process. It is an invariant **even for viscous flows** compared to other classical invariants, the Kelvin circulation invariant and the Helmholtz vorticity-flux invariant, which hold only for inviscid flows. That the force-field F_p is a conservative field means that if we take two points $x_1(0)$, $x_2(0)$, and any one-dimensional path $P(x_1(0), x_2(0))$, starting and ending on them, then for any test particle of mass m, the integral of the work done by the forces is independent of the particular path and depends only on the two points $x_1=x_1(0)$ and $x_2=x_2(0)$, and we denote it here by $W(x_1,x_2)$.

$$W(x_1, x_2) = \int_{P(x1,x2)} F_p ds \qquad (13)$$

In particular, it is known by the **gradient theorem** that this work is equal to the potential difference at these points, in this case the pressure:

$$W(x_1, x_2)=(1/c)||p(x_2(0))-p(x_1(0))||. \qquad (14)$$

Here, the constant (1/c) is set because of the normalization of the constant density in the Navier-Stokes equations, and to account for the correct dimensions of the measurement units for pressure, force and work.

Similarly, if we take a test-flow with test particles instead of one test particle, in the limit of points, the working density again depends only on the two points x_1 and x_2. In the next arguments, we will not use path invariance. Based on the hypothesis that the pressure is uniformly bounded in time by the same constant M in [0,T*), we deduce that the integral on the trajectory in Equation13 is also uniformly bounded in time by the same constant M in [0,T*).

$$\int_{P(x0,x1)} F_p ds <= M$$

We may rewrite this integral by changing the integral parameter into time as

$\int_{P(t0,t1)} F_p \frac{ds}{dt} dt <= M$ in [0,T*). Or as it is on the trajectory

$$\int_{P(t0,t1)} F_p u_m dt <= M \text{ in } [0,T^*). \quad (15)$$

where u_m is the (material) velocity on the trajectory.

If u_m->+∞ blows up as t->T*, then the material or convective acceleration is also

$\frac{Du_m}{Dt}$ ->+∞, which will blow up as t->T*,

Besides, from the Navier-Stokes equations on the trajectory path

$$\frac{Du}{Dt} = -\nabla p + \nu\Delta u_i \quad (16)$$

The pressure forces $F_p = -\nabla p$ -> +∞ will blow up as t->T*, since the friction term is subtracted from the pressure forces only.

Nevertheless, if both F_p and u_m blow up, then the integral in Equation 15 is also contradictory. Thus the (material) velocities do not blow up! Therefore, we can apply the necessary and sufficient conditions for regularity as in Proposition 3.6 (Fefferman, 2006; **necessary and sufficient conditions for regularity in velocities)** and derive the regularity, QED.

THE FINITE ENERGY BOUNDED PRESSURE VARIANCE THEOREM FOR INVISCID AND VISCOUS FLOWS AND THE SOLUTION OF THE 4TH CLAY MILLENNIUM PROBLEM

Remark 5.1: This paragraph utilizes two simple techniques:

a) Energy conservation in various alternative forms and formulae.
b) The property of pressure forces being conserved in the present situation of incompressible flows (gradient theorem).

The Clay Millennium problem is not just a challenging exercise in mathematical calculations, but is an issue of standard modelling of physical reality, so we may utilize all our knowledge of the underlying physical reality.

In the strategy adopted in this paper to solve the 4th Clay Millennium problem, we will involve, in a short and elegant way, as much as possible the intuitive physical ideas that may lead us to choose the correct and successful mathematical formulae and techniques, still everything will be within strict and precise mathematical limits. As Tao (2014) remarked in his discussion of the Clay Millennium problem, it seems hopeless to prove that the velocity always remains bounded (regularity), by following the solution in the general case, due to the vast number of flow-solution cases. Moreover, the energy conservation does not help much. Besides, it seems to be so! However, we need smarter and faster ideas via the invariants of the flow. In particular, we need clever techniques to calculate part of the flow energy in alternative ways, using virtual-test flows, and alternative integrals of the virtual work of the pressure forces on instantaneous paths, and still having the dimensions of the physical units of energy. We will develop strategies based on the following factors:

(1) The conservation of energy and the hypothesis of finite initial energy. Then as in Proposition 3.5, we obtain from this necessary and sufficient condition of regularity that the partial derivatives of the Jacobean are bounded: $\|\nabla u(x,t)\|_{L_\infty} \leq M$, and are uniformly bounded within the maximum time interval [0, T*); that is, there exists a solution, then we need to highlight a formula that computes *the partial derivatives of the velocities from the integrals of the velocities* in space and time up to date, since the bounded energy invariant is in the form of integrals of velocities.

(2) The shortcut of physical magnitude with energy dimensions of physical units as actual computational energy: In other words, if we reach an expression in the calculations that has physical magnitude with physical dimensional units of energy then that expression does calculate energy. **This is very valuable and discriminates the solution of the problem as a mathematical-physical problem compared to solving it as a purely mathematical problem in the context of partial differential equations that might e.g. admit non-uniqueness of the solutions of the Navier-Stokes equations.**

(3) The technique of virtual-test flows on instantaneous paths to find special formulas for calculating energy from alternative magnitudes. Instead of having to recalculate the energy starting from the classical velocity-based formulae and transforming it as the fluid flows, we may use shortcuts to calculate part of the energy of the fluid based on alternative perceptions, such as virtual test-particle flows, and the work of the pressure forces on instantaneous paths. Of course, the alternative formulae must always have the physical units of energy dimensions.

(4) Meanwhile, one good idea to start with is to consider alternative ways of measuring forms of energy and their projections onto a bundle of paths even at a single moment in time and state of the fluid, and to relate it to its total energy being finite and remaining bounded throughout the flow. Such alternative measurements of parts of the energy projected onto a bundle of paths can

be done by integrating the conservative pressure forces F_p of the fluid (pressure gradient) on the spatial path AB and relating their resulting theoretical work to the pressure differences p(A)-p(B) they have, since the pressure is a potential factor for such conservative pressure forces.

PROPOSITION 5.1 (Finite energy, uniform time-bounded pressure variance, theorem)

Let a local time, t in [0,T), smooth flow solution with velocity u(x,t) and pressure p(x,t) of the Navier-Stokes equation for viscous fluids or the Euler equation of inviscid fluids have smooth Schwartz initial data and finite initial energy E(0), as in the standard formulation of the 4th Clay Millennium problem. Then the pressure differences $|p(x_2(t))-p(x_1(t))|$ *for any two points* $x_1(t)$ *and* $x_2(t)$, *for times when the solution exists, remain bounded by kE(0), where k is a constant depending on the initial conditions and E(0) is the finite initial energy.*

Proof: Let's look again at the Navier-Stokes equations, as in the initial formulation by Fefferman (2006), which we brought here

$$\frac{Du}{Dt} = -\nabla p + \nu \Delta u_i \qquad (17)$$

where $\frac{Du}{Dt}$ is the acceleration of the material along the trajectory path (The reader is reminded that in the Navier-Stokes equations, as in the case of density in Equation 1, it is constant and customary to either normalize to 1 or divide from the left side to include in the pressure and viscosity coefficients).

We can separate the forces (or forces multiplied by a constant mass density) acting on a point by the two terms on the right hand side as

$$F_p = -\nabla p \qquad (18)$$

which is the force-field due to the pressure and

$$F_v = \nu \Delta u \qquad (19)$$

which is the force-field due to viscosity.

We notice that Equation 14 defines a ***conservative force-field***, since it is the gradient of a scalar field, where the pressure, p, acts as a scalar potential. Moreover, this property, which is a **conservative force-field**, is **an invariant** during the flow. It is an invariant **even for viscous flows** in contrast to other classical invariants, the Kelvin circulation invariant and the Helmholtz vorticity-flux invariant, which hold only for inviscid flows. That the force-field F_p is a conservative field means that if we take two points $x_1(0)$ and $x_2(0)$ and any one-dimensional path $P(x_1(0), x_2(0))$ starting and ending on them, then for any test particle of mass m, the integral of the work done by the forces is independent of the particular path and depends only on the two points $x_1=x_1(0)$ and $x_2=x_2(0)$, and we denote it here by $W(x_1,x_2)$.

$$W(x_1, x_2) = \int_{P(x1,x2)} F_p ds \qquad (20)$$

In particular, it is known by the **gradient theorem** that this work is equal to the potential difference at these points, in this case the pressure:

$$W(x_1, x_2)=(1/c)||p(x_2(0))-p(x_1(0))|. \qquad (21)$$

Here, the constant (1/c) is set because of the normalization of the constant density in the Navier-Stokes equations, and to account for the correct dimensions of the measurement units for pressure, force, and work.

Similarly, if we take a test-flow with test particles instead of one test particle, in the limit of points, the working density again depends only on the two points x_1 and x_2.

Now let again the two points $x_1(0)$ and $x_2(0)$ be in the initial conditions of the flow, and then when we assume Schwartz smooth initial conditions (and do not connect the compact smooth initial conditions), there is at least one double circular cone denoted by $DC(x_1(0),x_2(0))$, consisting of two circular cones united at their circular base C, whose vertices $x_1(0)$ and $x_2(0)$ are opposite to the plane of the common circular base C. Then, we adopt a bundle of paths starting at $x_1(0)$ and ending at $x_2(0)$ and filling all the double cones DC. We can now assume that a test-fluid (a flow of test-particles), inside this double cone of volume V, flows from $x_1(0)$ to $x_2(0)$ along these paths. Now let us integrate the work density along the paths as the pressure forces F_p of the original fluid acts on the test-fluid and inside this 3-dimensional double cone $DC(x_1(0),x_2(0))$. **This will give an instance of a spatial distribution of the work done by the pressure forces in the fluid, as projected onto assumed paths**. This energy comes from the immediate action of the spatially distributed pressure forces and **depends not only on the volume of integration, but also on the chosen path bundle**. It is a double integral, one- and two-dimensional (e.g. at the point of the circular base C), covering all the interior of the double cone DC. Since the working density per path is constant for each such path, by utilizing Fubini's theorem (SPIVAK, 1965), the final integral is:

$$W = \int_C \int_{x1}^{x2} F_p dx ds$$

$$W(0) = c \cdot V \cdot || p(x2(0)) - p(x1(0))| . \qquad (22)$$

On the other hand, this work that would be done by the pressure forces of the original fluid at any time *t* is the real energy. It is **an instance of a spatial distribution of the work done by the pressure forces in the fluid, as projected onto assumed paths, which** would be subtracted from the finite initial energy E(0). **Although this energy is only an instance at a fixed time *t* of the spatial distribution of the action of the pressure forces, as projected onto the assumed path bundle, it still has to be finite, as calculated in the 3-dimesional double cone. Therefore, this translates into a time instance in which the energy flow of the original fluid due to pressure forces projected onto the assumed path bundle is uniformly bounded in time, or in other words, bounded in each finite time interval.** Therefore:

$$W<=E(0). \qquad (23)$$

Combining Equation 23 with Equation 22, we obtain

$$c \cdot V \cdot \| \text{p(x2(0))} - \text{p(x1(0))} | \leq \text{E(0)} \qquad (24)$$

As we remarked, since the force field F_d induced by the pressure is conservative and invariant to the flow, as is the volume, we can repeat this argument for later times *t* in [0,T), and thus we also have

$$W(t) = c \cdot V \cdot \| \text{p(x2(t))} - \text{p(x1(t))} | \leq \text{E(t)} \qquad (25)$$

However, due to energy conservation, we have E(t)<=E(0) (for inviscid fluids E(t)=E(0)), then it also holds

$$c \cdot V \cdot \| \text{p(x2(t))} - \text{p(x1(t))} | \leq \text{E(0)} \qquad (26)$$

which is all that is required to prove for x_1, x_2 and k=1/(cV).

In particular, we notice that if there exists a supremum sup(p) and an infimum inf(p) of pressure at time *t*, in which case | sup(p)- inf(p) | is a measure of the variance of the pressure at time *t*, then this variance is bounded by the initial finite energy to a constant, justifying the title of the theorem. For the case of a fluid with a smooth compact connected support initial data, the infimum of the pressure is zero, which occurs at the boundary of the compact support. ***So, the pressures in general, are uniformly bounded by the same constants throughout the time interval [0,T*) (which includes the case T*=+∞).*** QED.

PROPOSITION 5.2 (The solution of the Clay Millennium problem about the regularity of the Navier-Stokes equations)

Let a local time, t in [0,T), and smooth flow solution with velocities u(x,t) of the Navier-Stokes equations of viscous fluids has smooth Schwartz initial data and finite initial energy E(0), as in the standard formulation of the 4th Clay Millennium problem. Then the solution is regular; in other words, it can be extended to a smooth solution for all times t in [0,+∞).

Proof: From the previous Proposition 5.1, we obtain that the pressures are smooth and bounded for finite time intervals, thus we apply the pressure sufficient condition of regularity as in Proposition 4.1 **(The pressure sufficient condition for regularity).** Hence, the solution of the Clay Millennium problem appears in its original formulation. We have also shown not only that there is no blow-up at finite time, but also that there is no blow-up even at time= +∞ QED.

Epilogue

In this paper, the regularity of the Navier-Stokes equations has been proven in a shorter way and with smooth Schwartz initial values than the authors' previous solution to this Millennium problem in Kyritsis (2021a). ***Finite initial energy, conservation of energy and regularity of pressure in Poisson's equation that relate pressures to velocities in the Navier-Stokes equations finally give regularity of the Navier-Stokes equation with standard hypotheses for initial data in the Clay Millennium problem.***

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